* Linear Differential Equations – If the degree of the dependent variable and all derivatives is one, such differential equations are called linear differential equations e.g. –

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = x^2 + x + 1$$

$$2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 3x = f(t)$$

* Non Linear Differential Equations – If the degree of the dependent variable and / or its derivatives are greater than 1 such differential equations are called non-linear differential equations.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = \sin x$$

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = e^x$$

* Linear Differential Equations of Second Order with Constant Coefficients –

The general form of the linear differential equation of second order is –

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P and Q are constant and R is a function of x or constant.

ear Independence and dependence –

Two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if –

$$A(y_1)(x) + By_2(x) \neq 0$$

A and B are not equal to zero.

* Complete solution = Complementary Function + Particular Integral

$$y = C.F. + P.I.$$

* Method For Finding the Complementary function –

S.N.	Nature of Roots of A.E.	Roots	C.F.
1.	Real (rational) and Distinct roots	m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2.	Repeated roots	$m_1 = m_2$, $m_1 = m_2 = m_3$	$(c_1 + c_2 x) e^{m_1 x}$ $(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
3.	Complex roots	$m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$	$e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$
4.	Repeated Complex roots	$m_1 = m_2 = \alpha + i\beta$ $m_3 = m_4 = \alpha - i\beta$	$e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$
5.	Irrational roots	$m_1 = a + \sqrt{b}$ $m_2 = a - \sqrt{b}$	$e^{ax}[c_1 \cosh\sqrt{b}\,x + c_2 \sinh\sqrt{b}\,x]$

Repeated irrational roots	$m_1 = m_2 = a+\sqrt{b}$ $m_3 = m_4 = a-\sqrt{b}$	$e^{ax}[(c_1+c_2x)\cosh\sqrt{b}x + (c_3+c_4x)\sinh\sqrt{b}x]$

Case-I- 1. Ques-> Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Soln -> Put $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$$(D^2 - 5D + 6)y = 0$$

$$D^2 - 5D + 6 = 0$$

$$A.E. = m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$m = 2, 3$$

$$C.F. = c_1 e^{2x} + c_2 e^{3x}$$

and $P.I. = 0$

$$y = C.F. + P.I.$$

$$y = c_1 e^{2x} + c_2 e^{3x}$$

Ques-> 2. Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Soln -> $A.E. = m^3 - 6m^2 + 11m - 6 = 0$

$$(m-1)(m-2)(m-3) = 0$$

$$m = 1, 2, 3$$

$$C.F. = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

$$P.I. = 0$$

$$y = C.F. + P.I.$$

Case - II When the roots of Auxilliary eq^n are equal.

Ques-3.) Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Sol^n-) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$

$(D^2 - 6D + 9)y = 0$

$A.E. = m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$m_1 = 3$

$m_2 = 3$

$C.F. = (c_1 + c_2x)e^{3x}$

$P.I. = 0$

$y = C.F. + P.I$

$= (c_1 + c_2x)e^{3x}$

Ques-) 4) Solve $\frac{d^4y}{dx^4} - 7\frac{d^3y}{dx^3} + 15\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 4y = 0$

$m = 1, 1, 1, 4$

Ques-5-) Find the general sol^n of differential eq^n.

Sol^n-) Here, $D^5y - D^3y = 0$

$D^3(D^2 - 1)y = 0$

$D^3(D^2 - 1) = 0$

$A.E. = 0 \Rightarrow m^3(m^2 - 1) = 0$

$m = 0, 0, 0, 1, -1$

Case - II When the roots of Auxilliary eqⁿ are equal.

Ques-3.) Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Solⁿ-) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$

$(D^2 - 6D + 9)y = 0$

$A.E. = m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$m_1 = 3$

$m_2 = 3$

$C.F. = (c_1 + c_2 x)e^{3x}$

$P.I. = 0$

$y = C.F. + P.I.$

$= (c_1 + c_2 x)e^{3x}$

Ques-) 4) Solve $\frac{d^4y}{dx^4} - 7\frac{d^3y}{dx^3} + 15\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 4y = 0$

$m = 1, 1, 1, 4$

Ques-5-) Find the general solⁿ of differential eqⁿ.

Solⁿ-) Here, $D^5y - D^3y = 0$

$D^3(D^2 - 1)y = 0$

$D^3 = D^2 - 1) = 0$

$A.E. = 0 \Rightarrow m^3(m^2 - 1) = 0$

$m = 0, 0, 0, 1, -1$

Hence its solⁿ is $y = (c_1 + c_2 x + c_3 x^2) + c_4 e^{x}$
$+ c_5 e^{-x}$

Case III – when two roots of Auxilliary eqⁿ are complex.

6. Ques) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

Solⁿ) $(D^2 + 4D + 5)y = 0$

$A.E. = m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-1}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

Hence. $C.F. = e^{-2x}[c_1 \cos x + c_2 \sin x]$

Case IV – when roots of Auxiliary eqⁿ are repeated imaginary –

$m_1 = m_2 = \alpha + i\beta$
$m_3 = m_4 = \alpha - i\beta$

$C.F. = (c_1 + c_2 x)\, e^{(\alpha + i\beta)x} + (c_3 + c_4 x)\, e^{(\alpha - i\beta)x}$
$+ c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$

A.E. is $m^3 - 5m^2 + 5m - 1 = 0$

$$(m-1)(m^2 - 4m + 1) = 0$$

if $m - 1 = 0$

$m = 1$

or $m^2 - 4m + 1 = 0$

$$m = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

P.I. $= 0$

$$C.F. = c_1 e^{x} + e^{2x}\left[c_2 \cosh\sqrt{3}x + c_3 \sinh\sqrt{3}x\right]$$

$$y = c_1 e^{x} + e^{2x}\left(c_2 \cosh\sqrt{3}x + c_3 \sinh\sqrt{3}x\right]$$

Ans —

Case VI - When roots of Auxiliary eqn are repeated irrational.

$m_1 = m_2 = a + \sqrt{b}$

$m_3 = m_4 = a - \sqrt{b}$

$$C.F. = e^{ax}\left[(c_1 + c_2 x)\cosh\sqrt{b}x + (c_3 + c_4 x)\sinh\sqrt{b}x\right] + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

⑦

Ques-7-) Solve $(D^2+1)^2(D-1)y = 0$

Soln-) The auxiliary eqn is —

$$(m^2+1)^2(m-1)y = 0$$

$$m-1=0$$

$$\Rightarrow m=1$$

$$(m^2+1)^2 = 0$$

$$m^2+1=0$$

$$m^2=-1$$

$$m=\sqrt{-1}$$

$$=\pm i$$

$$m = 0 \pm i$$

$$\alpha = 0, \ \beta = 1$$

$$C.F. = c_1 e^{x} + [(c_2+c_3x)\cos x + (c_4+c_5x)\sin x]$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constant of integration.

Case v-) when roots of auxilliary equation are irrational.

Ques 8-) Solve $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - y = 0$

Soln-) The Given eqn is —

$$(D^3-5D^2+5D-1)y = 0$$

$$D^3-5D^2+5D-1=0$$

Ques 9-) Solve $(D^2-4D+1)^2 y = 0$

Soln-) we have $(D^2-4D+1)^2 y = 0$

Its auxiliary eqn is $(m^2-4m+1)^2 = 0$

$$m^2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

$$m = 2 \pm \sqrt{3},\ 2 \pm \sqrt{3}$$

These roots are repeated and irrational.

$$C.F. = e^{2x}\left[(c_1 + c_2 x)\cosh\sqrt{3}\,x + (c_3 + c_4 x)\sinh\sqrt{3}\,x\right]$$

$$P.I. = 0$$

$$y = C.F. + P.I.$$

Ques-10-) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Ans- $y = c_1 e^{x} + c_2 e^{2x}$

Ques. 11-) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

Ans- $y = (c_1 + c_2 x)\, e^{4x}$

Rules to find Particular integral –

1-) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

(i) If $f(a) = 0$ then the above rule fails.

Then $\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}$

$$\boxed{\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(a)} e^{ax}}$$

(ii) if $f'(a) = 0$ then $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax}$

2-) $\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$

Expand $[f(D)]^{-1}$ and then operate.

3-) $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ and $\frac{1}{f(D^2)} \cos ax$
$= \frac{1}{f(-a^2)} \cos ax$

i-) if $f(-a^2) = 0$ then $\frac{1}{f(D^2)} \sin ax = x \cdot \frac{1}{f'(-a^2)} \sin ax$

and $\frac{1}{f(D^2)} \cos ax = x \cdot \frac{1}{f'(-a^2)} \cos ax$

4) $\frac{1}{f(D)} e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)} \phi(x)$

5) $\frac{1}{D+a} \phi(x) = e^{-ax} \int e^{ax} \cdot \phi(x)\, dx$

1* $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Ques-12) Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$

Sol^n) Here we have –

$$(D^2 + 6D + 9)y = 5e^{3x}$$

Auxilliary eq^n is –

$$m^2 + 6m + 9 = 0$$

$$m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$(m+3)(m+3) = 0$$

$$(m+3)^2 = 0$$

$$m = -3, -3$$

$$C.F. = (c_1 + c_2 x)e^{-3x}$$

$$P.I. = \frac{1}{D^2 + 6D + 9} \cdot 5e^{3x}$$

$$= 5 \cdot \frac{1}{D^2 + 6D + 9} e^{3x}$$

$$= 5 \cdot \frac{1}{3^2 + 6 \times 3 + 9} e^{3x}$$

$$= 5 \cdot \frac{e^{3x}}{36}$$

$$= \frac{5}{36} e^{3x}.$$

The complete solution is —

$$y = (c_1 + c_2 x) e^{-3x} + \frac{5 e^{3x}}{36}$$

Ques-13-) Solve $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = e^{ax} + e^{bx}$

Sol^n-) $P.I. = \frac{1}{D^2 - (a+b)D + ab} e^{ax} + \frac{1}{D^2 - (a+b)D + ab} e^{bx}$

Ans- $y = c_1 e^{ax} + c_2 e^{bx} + \frac{x}{a-b}[e^{ax} - e^{bx}]$

Ques-14-) Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

Sol^n-) Here, $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

A.E. $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3, 3$

C.F. $= (C_1 + C_2 x) e^{3x}$

$$P.I. = \frac{1}{D^2 - 6D + 9} 6e^{3x} + \frac{1}{D^2 - 6D + 9} 7e^{-2x}$$

$$+ \frac{1}{D^2 - 6D + 9}(-\log 2)$$

$$= x \cdot \frac{1}{2D-6} 6e^{3x} + \frac{1}{4+12+9} 7e^{-2x} - \log 2 \frac{1}{D^2-6D+9} \cdot e^{0x}$$

↓

if $f(a) = 0$

$a = 3$

(10)

$$= x^2 \frac{1}{2} \cdot 6 \cdot e^{3x} + \frac{7}{25} e^{-2x} - \log 2 \left(\frac{1}{9}\right)$$

$$= 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$$

Complete solⁿ is $y = (c_1 + c_2 x) e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x}$
$- \frac{1}{9} \log 2$

✱2) $\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$

Ques-15) Find the particular integral of $\frac{d^2y}{dx^2} - y = x^2$

Solⁿ) We have $\frac{d^2y}{dx^2} - y = x^2$

$$(D^2 - 1) y = x^2$$

$$P.I. = \frac{1}{(D^2-1)} x^2$$

$$= -(1-D^2)^{-1} x^2$$

$$= -(1+D^2) x^2$$

$$= -(x^2+2)$$

$\therefore D^2 x^2$

$\frac{d^2x^2}{dx^2} = \frac{d\,2x}{dx}$

$= 2.$

(3) $\frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)}$

$\frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)}$

Ques16) Find the particular integral of $(D^2-4D+4)y = \cos 2x$

Soln)

$(D^2-4D+4)y = \cos 2x$

$P.I. = \frac{1}{D^2-4D+4} \cdot \cos 2x$

$= \frac{1}{2^2-4D+4} \cdot \cos 2x$

$= \frac{1}{-4D} \cos 2x$

$= -\frac{1}{4} \int \cos 2x \, dx$

$= -\frac{1}{4} \frac{\sin 2x}{2}$

$= -\frac{1}{8} \sin 2x$

Ans

$D \rightarrow$ Symbol D stands for the operation of differential

$\frac{1}{D} \rightarrow$ stands for the operation of integration

$\frac{1}{D^2} \rightarrow$ stands for the operation of integration twice.

Ques-17-> Solve the following differential equation.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$$

Solⁿ-> Here, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \sin 3x + \cos 2x$

$\Rightarrow (D^2 + 5D - 6)y = \sin 3x + \cos 2x$

A.E. is $m^2 + 5m - 6 = 0$

$(m-1)(m+6) = 0$

$m = 1, -6$

$C.F. = c_1 e^{x} + c_2 e^{-6x}$

$$P.I. = \frac{1}{D^2+5D-6}(\sin 3x + \cos 2x)$$

$$= \frac{\sin 3x}{D^2+5D-6} + \frac{\cos 2x}{D^2+5D-6}$$

$$= \frac{\sin 3x}{-9+5D-6} + \frac{\cos 2x}{-4+5D-6}$$

$$= \frac{\sin 3x}{5D-15} + \frac{\cos 2x}{5D-10}$$

$$= \frac{\sin 3x}{5(D-3)} + \frac{\cos 2x}{5(D-2)}$$

$$= \frac{(D+3)\sin 3x}{5.(D+3)(D-3)} + \frac{(D+2)\cos 2x}{5(D-2)(D+2)}$$

$$= \frac{(D+3)\sin 3x}{5(D^2-9)} + \frac{(D+2)\cos 2x}{5(D^2-4)}$$

(13)

$$= \frac{(D+3)\sin 3x}{5(D^2-9)} + \frac{(D+2)\cos 2x}{5(D^2-4)}$$

$$= \frac{(D+3)\sin 3x}{5(-9-9)} + \frac{(D+2)\cos 2x}{5(-4-4)}$$

$$= \frac{(D+3)\sin 3x}{-90} + \frac{(D+2)\cos 2x}{-40}$$

$$= -\frac{1}{90}(3\cos 3x + 3\sin 3x) - \frac{1}{40}(-2\sin 2x + 2\cos 2x)$$

$$= -\frac{1}{30}\cos 3x - \frac{1}{30}\sin 3x + \frac{1}{20}\sin 2x - \frac{1}{20}\cos 2x$$

$$= -\frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x).$$

Complete solution is -

$$y = C.F. + P.I.$$

$$y = c_1 e^{x} + c_2 e^{-6x} - \frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x)$$

Ques-18.) Solve $\frac{d^2y}{dx^2} + 4y = \sin^2 2x$ with condition

$y(0) = 0, \; y'(0) = 0$

Soln-) $(D^2+4)y = \sin^2 2x$

A.E. is $m^2+4 = 0$

$m^2 = -4$

$m = \pm 2i$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x$$

$$P.I. = \frac{1}{D^2+4} \sin^2 2x$$

$$= \frac{1}{2} \cdot \frac{(1-\cos 4x)}{D^2+4}$$

$$= \frac{1}{2} \cdot \frac{1}{D^2+4} - \frac{1}{2} \frac{\cos 4x}{(D^2+4)}$$

$\cos 4x = 2\cos^2 2x - 1$

$= 1 - 2\sin^2 2x$

$2\sin^2 2x = 1 - \cos 4x$

$\sin^2 2x = \frac{1}{2}(1-\cos 4x)$

$$= \frac{1}{2} \cdot \frac{e^{0 \cdot x}}{-(0)^2+4} - \frac{1}{2} \cdot \frac{\cos 4x}{(-4^2+4)}$$

$$= \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{\cos 4x}{-16+4}$$

$$= \frac{1}{8} + \frac{\cos 4x}{24}$$

complete solution is $y = C.F. + P.I.$

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8} + \frac{1}{24} \cos 4x \quad \text{----(i)}$$

On putting $x=0$ and $y=0$ in (i) we get –

$$0 = C_1 \cdot 1 + 0 \cdot C_2 + \frac{1}{8} + \frac{1}{24}$$

$$0 = \frac{3+1}{24} + C_1$$

$$\boxed{-\frac{1}{6} = C_1}$$

Differentiating eqn (i) we get –

$$y'(x) = -2C_1 \sin 2x + 2C_2 \cos 2x - \frac{1}{6} \sin 4x \quad \text{---(2)}$$

On putting $x=0$, $y'(x)=0$ in (2) we get –

(15)

$$0 = 2c_2 \Rightarrow c_2 = 0$$

On putting the values of c_1 and c_2 in (i) we get –

$$y = -\frac{1}{6}\cos 2x + \frac{1}{8} + \frac{1}{24}\cos 4x$$

Ques-19) Solve $\frac{d^2y}{dx^2} + y = \sin x \sin 2x$

Sol^n -> We have –

$$\frac{d^2x}{dx^2} + y = \sin x \cdot \sin 2x$$

$$(D^2+1)y = \sin x \cdot \sin 2x$$

A.E. is $m^2 + 1 = 0$

$$\boxed{m = \pm i}$$

$$C.F. = c_1 \cos x + c_2 \sin x$$

$$P.I. = \frac{1}{D^2+1} \sin x \cdot \sin 2x$$

$$= \frac{1}{D^2+1} \cdot \frac{1}{2}(2\sin x \cdot \sin 2x)$$

$$= \frac{1}{2} \cdot \frac{1}{D^2+1} (\cos x - \cos 3x)$$

$$= \frac{1}{2} \cdot \left[\frac{\cos x}{D^2+1} - \frac{\cos 3x}{D^2+1}\right]$$

$$= \frac{1}{2}\left[x \cdot \frac{1}{2D}\cos x - \frac{1}{-9+1}\cos 3x\right]$$

$\therefore$ Diff^n.

$$= \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{16}\left[4x \sin x + \cos 3x\right]$$

Complete Sol^n is $y = C.F. + P.I.$

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{16}(4x \sin x + \cos 3x)$$

$$\frac{1}{f(D)} \cdot e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)} \cdot \phi(x)$$

Ques-20) Solve $(D^4-1)y = e^x \cos x$

Soln-) Here we have, $(D^4-1)y = e^x \cos x$

A.E. is $m^4 - 1 = 0$

$$(m+1)(m-1)(m^2+1) = 0$$

$$m = 1, -1, +i, -i$$

$$C.F. = c_1 e^{-x} + c_2 e^x + (c_3 \cos x + c_4 \sin x)$$

$$P.I. = \frac{1}{D^4-1} e^x \cos x$$

$$= e^x \cdot \frac{1}{(D+1)^4 - 1} \cos x$$

(by above rule)

$$= e^x \cdot \frac{1}{D^4 + 4D^3 + 4D^2 + 4D} \cdot \cos x$$

$$= e^x \cdot \frac{1}{(-1)^2 + 4 \cdot D \cdot (-1) - 4 + 4D} \cos x$$

$$= e^x \cdot \frac{1}{1-4} \cos x$$

$$= -\frac{1}{3} e^x \cos x$$

Complete soln is $y = C.F. + P.I.$

$$y = c_1 e^{-x} + c_2 e^x + (c_3 \cos x + c_4 \sin x) - \frac{1}{3} e^x \cdot \cos x$$

Ques-21-) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \cdot e^x \cos x$

Solⁿ-> Here we have, $(D^2 - 2D + 1)y = x \cdot e^x \cos x$

A.E. is $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$m = 1, 1$

$C.F. = (c_1 + c_2 x)e^x$

$$P.I. = \frac{1}{D^2 - 2D + 1} x \cdot e^x \cos x$$

$$= \frac{1}{(D-1)^2} \cdot x \cdot e^x \cos x$$

$$= \frac{1}{(D-1+1)^2} \cdot x \cdot e^x \cos x$$

$$= \frac{1}{D^2} \cdot e^x \cdot x \cos x$$

$$= e^x \cdot \frac{1}{D}[x \cdot \sin x - (1)(-\cos x)]$$

$$= e^x \cdot \frac{1}{D}(x \sin x + \cos x]$$

$$= e^x [x(-\cos x) - 1(-\sin x) + \sin x]$$

$$= e^x[-x\cos x + \sin x + \sin x]$$

$$= e^x[-x\cos x + 2\sin x]$$

$$= e^x[2\sin x - x\cos x]$$

$y = C.F. + P.I.$

Ques-22-> Solve the differential equation $(D^2-4D+4)y = 8x^2 . e^{2x} . \sin 2x$

Soln-> Here we have. $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$

A.E. is $(m^2-4m+4)=0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$C.F. = (c_1 + c_2 x)e^{2x}$$

$$P.I. = \frac{1}{D^2-4D+4} 8x^2 . e^{2x} . \sin 2x$$

$$= 8 . \frac{1}{(D-2)^2} x^2 . e^{2x} \sin 2x$$

$$= 8 . e^{2x} . \frac{1}{(D-2+2)^2} x^2 \sin 2x$$

(By above rule)

$$= 8 e^{2x} \frac{1}{D^2} \underset{I}{x^2} \underset{II}{\sin 2x}$$

$$= 8 e^{2x} . \frac{1}{D} \left[x^2 \left(-\frac{\cos 2x}{2}\right) - 2x\left(-\frac{\sin 2x}{4}\right) + 2 . \left(\frac{\cos 2x}{8}\right)\right]$$

$$= 8 e^{2x} . \frac{1}{D} \left[-\frac{x^2}{2} \cos 2x + \frac{x . \sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 8 e^{2x} \left[-\frac{x^2}{2}\left(\frac{\sin 2x}{2}\right) - \left(\frac{-2x}{2}\right)\left(-\frac{\cos 2x}{4}\right) + (-1)\left(-\frac{\sin 2x}{8}\right) + \frac{x}{2}\left(-\frac{\cos 2x}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sin 2x}{4}\right) + \frac{\sin 2x}{8}\right]$$

$$= e^{2x}\left[-2x^2\sin 2x - 2x\cos 2x + \sin 2x - 2x\cos 2x + \sin 2x + \sin 2x\right]$$

$$= e^{2x}\left[-2x^2\sin 2x - 4x\cos 2x + 3\sin 2x\right]$$

$$= -e^{2x}\left[4x\cos 2x + (2x^2-3)\sin 2x\right]$$

$$Y = C.F. + P.I.$$

$$= (c_1 + c_2 x)e^{2x} - e^{2x}\left[4x\cos 2x + (2x^2-3)\sin 2x\right]$$

Ans

⊛ To Find the value of $\frac{1}{f(D)} x^n \sin ax$.

Now $\frac{1}{f(D)} x^n (\cos ax + i\sin ax) = \frac{1}{f(D)} x^n e^{iax}$

$$= e^{iax} \cdot \frac{1}{f(D+ia)} x^n$$

$$\boxed{\frac{1}{f(D)} \cdot x^n \sin ax = \text{Imaginary part of } e^{iax} \cdot \frac{1}{f(D+ia)} \cdot x^n}$$

$$\boxed{\frac{1}{f(D)} \cdot x^n \cos ax = \text{Real part of } e^{iax} \cdot \frac{1}{f(D+ia)} x^n}$$

Ques-23-) Solve the differential equation

$$(D^2+2D+1)y = x\cos x$$

Soln-) Here we have —

$$(D^2+2D+1)y = x\cos x$$

A.E. is —

$$m^2+2m+1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

$$C.F. = (c_1 + c_2 x)e^{-x}$$

$$P.I. = \frac{1}{(D+1)^2} x \cdot \cos x$$

$$= \text{Real Part of } e^{iax} \cdot \frac{1}{f(D+ia)} \cdot x^n$$

$$a = 1$$

$$= \text{Real part of } \frac{1}{(D+1)^2} x \cdot (\cos x + i\sin x)$$

$$= \text{Real part of } \frac{1}{(D+1)^2} x \cdot e^{ix}$$

$$= \text{Real part of } e^{ix} \cdot \frac{1}{(D+i+1)^2} \cdot x$$

$$= \text{Real part of } e^{ix} \cdot \frac{1}{D^2+2(i+1)D+(i+1)^2} \cdot x$$

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$$= \text{Real part of } e^{ix} \cdot \frac{1}{D^2+2(1+i)D+2i} x$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\left[\frac{1}{1+\frac{1+i}{i}D+\frac{D^2}{2i}} x\right]$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\left[1+\frac{1+i}{i}D+\frac{D^2}{2i}\right]^{-1} \cdot x$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\left[1-\left(\frac{1+i}{i}\right)D+ \cdots\right] x$$

$$= \text{Real part of } \frac{1}{2i}(\cos x + i\sin x)\left[x - \frac{1+i}{i}\right]$$

$$= \text{Real part of } -\frac{i}{2}(\cos x + i\sin x)(x+i-1)$$

$$= \text{Real part of } -\frac{1}{2}(i\cos x - \sin x)(x+i-1)$$

$$= \frac{1}{2}(\sin x)(x-1) + \frac{1}{2}\cos x$$

$y =$ C.F. + P.I.

$$= (c_1 + c_2 x)e^{-x} + \frac{1}{2}(x-1)\sin x + \frac{1}{2}\cos x$$

Ans