MATHEMATIS-I — Module-V

* KAS-103 T * — * Vector Calculus *

* Lecture No - 42 * [redacted]

Introduction of vector, scalar funcⁿ and basic operation →

(*) **Vector** → A vector is a quantity which has magnitude and direction.

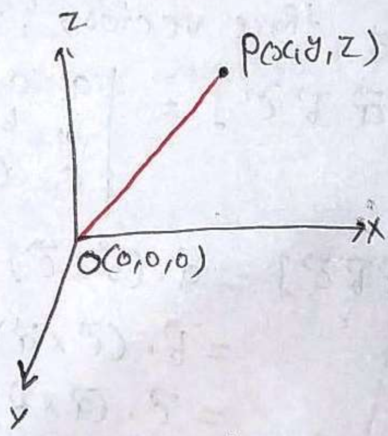
Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a given vector with initial point $P(x_1, y_1, z_1)$ & terminal point $Q(x_2, y_2, z_2)$. Then three coordinates difference b/w P & Q are

$$a_1 = x_2 - x_1 \;,\; a_2 = y_2 - y_1 \;,\; a_3 = z_2 - z_1$$

are the components of a vector $\vec{a}$ and length or magnitude of vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

(*) **Position vector** →



Let $\vec{r} = \vec{OP}$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

is a vector of a point $P(x, y, z)$ is the vector with initial point $O(0,0,0)$ & terminal point $P(x, y, z)$.

Then $\boxed{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$ is called position vector.

& $\boxed{r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}}$ → Magnitude of position vector.

(*) **Operations on vectors** →

① **Dot or Scalar product** →

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then ① $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

② $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

③ $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Dot product of two vectors is allways a scalar quantity.

② Cross product or Vector product→

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then

(i) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(ii) $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta$.

(iii) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Cross product of two vectors is always a vector quantity.

③ Scalar Triple Product→

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be any three vectors, then scalar triple product is

(i) $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(ii) $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b})\cdot\vec{c}$

$= \vec{b}\cdot(\vec{c} \times \vec{a}) = (\vec{b} \times \vec{c})\cdot\vec{a}$

$= \vec{c}\cdot(\vec{a} \times \vec{b}) = (\vec{c} \times \vec{a})\cdot\vec{b}$.

(iii) If $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ then vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

(iv) It gives scalar quantity.

④ Vector triple Product→

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

⑤ Unit Vector→

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

then unit vector $\boxed{\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}}$.

Scalar funcⁿ →

Scalar funcⁿ $f(x,y,z)$ is a funcⁿ defined at each point in a domain D. ~~Its value at each point in a domain D~~
Its value is real and depends only on the point $P(x,y,z)$.

<u>Ex</u> → (i) $f(x,y,z) = xyz^2$

(ii) $\phi(x,y,z) = x+y+z$.

Vector funcⁿ →

If to each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is called a vector funcⁿ of scalar variable t.

we write $\vec{r} = \vec{r}(t)$

$$= \vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}.$$

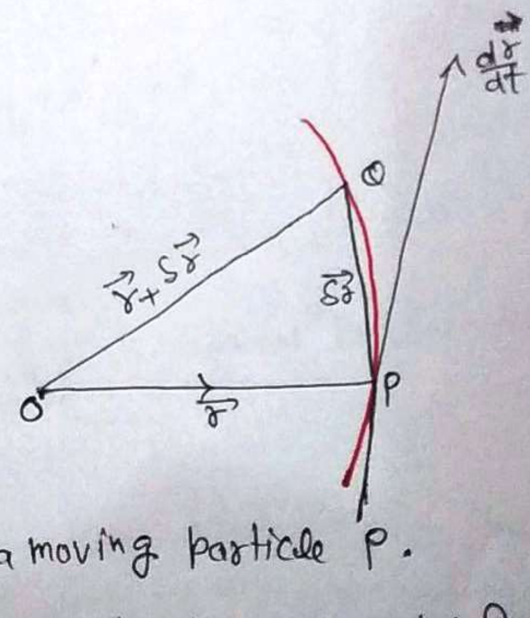
Derivative of a vector funcⁿ w.r.t scalar →

Let $\vec{r} = \vec{f}(t)$ be a vector funcⁿ of scalar variable t.
Then derivative of vector $\vec{r}$ w.r.t t is

$$\frac{d\vec{r}}{dt}.$$

Geometrical interpretation →

Hence $\frac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at point P.



<u>Note</u> → (i) $\vec{r}$ → Position of a vector of a moving particle P.

(ii) $\vec{v} = \frac{d\vec{r}}{dt}$ = Velocity vector of the particle at point P.

(iii) $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ = Acceleration of the particle at P.

General rules for differentiation:

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are vector funcⁿ of a scalar t and ϕ is a scalar funcⁿ of t. Then

(i) $\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$

(ii) $\frac{d}{dt}(\vec{a} \cdot \vec{b}) = \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b}$

(iii) $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$

(iv) $\frac{d}{dt}(\phi \vec{a}) = \phi \frac{d\vec{a}}{dt} + \frac{d\phi}{dt} \vec{a}$.

(v) $\frac{d}{dt}[\vec{a}\,\vec{b}\,\vec{c}] = \left[\frac{d\vec{a}}{dt}\,\vec{b}\,\vec{c}\right] + \left[\vec{a}\,\frac{d\vec{b}}{dt}\,\vec{c}\right] + \left[\vec{a}\,\vec{b}\,\frac{d\vec{c}}{dt}\right]$

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Scalar point func^n →
Let R be the region of space at each point of which a scalar point func^n $\phi = \phi(x,y,z)$ is given then ϕ is called a scalar point func^n and R is called a scalar field.

Ex → $\boxed{\phi(x,y,z) = x^2+y^2+z^2}$

Vector Point func^n → Let R be the region of space at each point of which a vector $\vec{v} = \vec{v}(x,y,z)$ is given, then $\vec{v}$ is called a vector point func^n and R is called a vector field.

Ex → The velocity of a moving fluid at any ~~int~~ instant is the vector point func^n.

Gradient of a scalar field →
Let $f(x,y,z)$ is a scalar point func^n, then gradient of f is defined by

$$\boxed{\begin{aligned}\text{grad } f = \nabla f &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)f \\ &= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\end{aligned}}$$

where vector operator
$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$ is called Del (or Nebla) operator.

Properties of gradient →
If f and g are two scalar point functions then

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(f \cdot g) = f\nabla g + \nabla f g$

(iii) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}, g \neq 0$

(iv) $\nabla(cf) = c\nabla f$

(v) Gradient of a constant function is allways zero
i.e $\nabla C = 0$

(vi) Gradient of ϕ is a vector quantity.

Ex-1: find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point $(1,-2,-1)$.

Solⁿ: Given $\phi = 3x^2y - y^3z^2$

Then $\text{grad}\,\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z).$$

At $(1,-2,-1)$

$$\nabla\phi = -12\hat{i} - 9\hat{j} - 16\hat{k}. \quad (\text{Put } x=1, y=-2, z=-1).$$

Ex-2: find grad f where

(i) $f = x^2 + yz$ at $(1,1,1)$. (Ans $2\hat{i}+\hat{j}+\hat{k}$)

(ii) $f = \log(x^2+y^2+z^2)$ [Ans $\frac{2}{3}(\hat{i}+\hat{j}+\hat{k})$].

Ex-2: If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, prove that grad u, grad v & grad w are coplanar vectors. (AKTU- 2005, 2010, 2015).

Solⁿ(2): Given

$$u = x+y+z$$
$$v = x^2+y^2+z^2$$
$$w = yz+zx+xy$$

Now $\text{grad}\,u = \nabla u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z} = \hat{i}\cdot 1 + \hat{j}\cdot 1 + \hat{k}\cdot 1$

$\text{grad}\,v = \nabla v = \hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\text{grad}\,w = \nabla w = \hat{i}\frac{\partial w}{\partial x} + \hat{j}\frac{\partial w}{\partial y} + \hat{k}\frac{\partial w}{\partial z} = (z+y)\hat{i} + (z+x)\hat{j} + \hat{k}(x+y).$

Now

$$[\text{grad}\,u, \text{grad}\,v, \text{grad}\,w] = \text{grad}\,u\cdot(\text{grad}\,v \times \text{grad}\,w)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} \quad \text{by } R_2 \to R_2 + R_3$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

$\boxed{\text{Hence grad u, grad v \& grad w are coplanar vectors.}}$

Note→ (i) $\boxed{\text{Unit normal vector to the surface } \hat{N} = \frac{\nabla\phi}{|\nabla\phi|}}$

Ex→1→ find a unit vector normal to the surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$. (AKTU-2014)

Sol^n→ Let $\phi = x^3+y^3+3xyz-3$, then $\frac{\partial\phi}{\partial x} = 3x^2+3yz$

$$\frac{\partial\phi}{\partial y} = 3y^2+3xz$$

$$\frac{\partial\phi}{\partial z} = 3xy.$$

Now $\nabla\phi = \text{grad}\,\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= (3x^2+3yz)\hat{i} + (3y^2+3xz)\hat{j} + 3xy\,\hat{k}$$

At $(1,2,-1)$, $\nabla\phi = -3\hat{i}+9\hat{j}+6\hat{k}$.

which is a vector normal to the given surface at $(1,2,-1)$.

Hence a unit vector normal to the given surface at $(1,2,-1)$

is $\hat{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{(-3)^2+9^2+6^2}} = \frac{1}{\sqrt{14}}(-\hat{i}+3\hat{j}+2\hat{k})$.

Ex-2→ Find a unit vector normal to the surface $x^2y+2xz=4$ at the point $(2,-2,3)$. (AKTU-2015). (Ans → $\frac{-\hat{i}+2\hat{j}+2\hat{k}}{3}$

Ex-3→ find a unit vector normal to the surface $xy^3z^2=4$ at $(-1,-1,2)$. Ans → $-\frac{1}{\sqrt{11}}(\hat{i}+3\hat{j}-\hat{k}$

$\boxed{\text{(Ex. for Position Vector } \vec{r}\text{)}}$

Ex-1→ If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $r = |\vec{r}|$ then prove that

(i) $\nabla f(r) = f'(r)\cdot\frac{\vec{r}}{r}$

(ii) $\nabla r^n = nr^{n-2}\vec{r}$

(iii) $\nabla \log r^n = \frac{n\vec{r}}{r^2}$

(iv) $\nabla \log r = \frac{\vec{r}}{r^2}$

(v) $\nabla e^{r^2} = 2e^{r^2}\vec{r}$

(vi) Prove $\text{grad} f(r) \times \vec{r} = \vec{0}$.

(vii) find $\nabla\left(\frac{1}{r}\right)$.

Sol^n→ Given $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

then $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ or $r^2 = x^2+y^2+z^2$ $\Big\}\to(1)$

From $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly

$\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$ $\Big\}\to(2)$.

Sol^n; (i) Now grad $f(r) = \nabla f(r)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \hat{i}\frac{\partial}{\partial x} f(r) + \hat{j}\frac{\partial}{\partial y} f(r) + \hat{k}\frac{\partial}{\partial z} f(r)$$

$$= \hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] \quad \text{(from 2)}$$

$$= \frac{f'(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= f'(r)\frac{\vec{r}}{r} \longrightarrow (3)$$

Sol^n: (vi) ~~Now~~

$$\text{grad} f(r) \times \vec{r} = f'(r)\frac{\vec{r}}{r} \times \vec{r} \quad \text{(from 3)}$$

$$= \frac{f'(r)}{r}(\vec{r} \times \vec{r})$$

$$= \frac{f'(r)}{r} \cdot 0 = 0.$$

(ii) $$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \hat{i}\frac{\partial}{\partial x} r^n + \hat{j}\frac{\partial}{\partial y} r^n + \hat{k}\frac{\partial}{\partial z} r^n$$

$$= \hat{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] \quad \text{(from 2)}$$

$$= n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow \boxed{\nabla r^n = n r^{n-2}\vec{r}} \longrightarrow (4)$$

(vii) Put $n = -1$ in eq^n (4), we get

$$\nabla r^{-1} = \nabla \frac{1}{r} = (-1) r^{-3}\vec{r}$$

$$= -\frac{\vec{r}}{r^3}.$$

Directional derivative →

The directional derivative of a scalar field f at a point $P(x,y,z)$ in the direction of unit vector $\hat{a}$ is

$$\boxed{\frac{df}{ds} = (\text{grad} f)\cdot\hat{a} \quad \text{or} \quad \frac{df}{ds} = \nabla f\cdot\hat{a}}$$

Note→ The maximum rate of increase of ϕ is $|\nabla\phi|$

or

The maximum value of directional derivative is $|\nabla\phi|$.

Ex-1→ Find the directional derivative of the funcn $f(x,y,z) = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of the vector $\hat{i}+2\hat{j}+3\hat{k}$.

Soln Given $f = xy^2+yz^3$

then $\nabla f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z} = \hat{i}\,y^2+\hat{j}(2xy+z^3)+\hat{k}(3yz^2)$

At $(2,-1,1)$, $\nabla f = \hat{i}-3\hat{j}-3\hat{k} \longrightarrow (2)$

Let $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ then

unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}(\hat{i}+2\hat{j}+3\hat{k}).$$

Hence directional derivative of f in the direction of $\vec{a}$

$$\text{is} = (\nabla f)\cdot\hat{a}$$
$$= (\hat{i}-3\hat{j}-3\hat{k})\cdot\frac{(\hat{i}+2\hat{j}+3\hat{k})}{\sqrt{14}} = \frac{1-6-9}{\sqrt{14}} = \frac{-14}{\sqrt{14}}$$
$$= -\sqrt{14}.$$

Ex-2→ Find the directional derivative of the funcn

(i) $f(x,y,z) = 2xy+z^2$ at the point $(1,-1,3)$ in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$. Ans $\frac{14}{3}$.

(ii) $\phi = 4xz^3-3x^2yz^2$ at $(2,-1,2)$ along z-axis. (Ans → 144.)

[Hint along z-axis $\vec{a} = 0\hat{i}+0\hat{j}+1\hat{k}$.

Ex-3→ Find the directional derivative of the funcn $f = x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. (AKTU-2009, 2011)

In what direction it will be maximum? Find magnitude also.

Sol^n We have $f = x^2 - y^2 + 2z^2$

Then $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$= 2\hat{i} - 4\hat{j} + 12\hat{k}$ at $P(1,2,3)$.

Also $\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 4\hat{i} - 2\hat{j} + \hat{k}$.

Let $\vec{a} = 4\hat{i} - 2\hat{j} + \hat{k}$ then $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4^2 + (2)^2 + 1^2}}$

$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$.

∴ Directional derivative of f in the direction $\vec{PQ} = (\nabla f) \cdot \hat{a}$

$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$

$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$.

The directional derivative of f is max in the direction of normal i.e $\nabla f = 2\hat{i} - 4\hat{j} + 2\hat{k}$.

The maximum value of this directional derivative $= |\nabla f|$

$= \sqrt{2^2 + (-4)^2 + 12^2} = \sqrt{164}$.

Ex-4) Find directional derivative of $\Psi = 4e^{x+5y-13z}$ at the point (1,2,3) in the direction towards the point (-3,5,7)

[Ans $- 4\sqrt{41}\, e^{-28}$]

Ex-5) find the directional derivative of $\phi = (x^2 + y^2 + z^2)^{-1/2}$ at the point P(3,1,2) in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$. (AKTU- 2014, 2017).

Sol^n Given $\phi = (x^2 + y^2 + z^2)^{-1/2}$

then

$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \hat{i}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x\right\} + \hat{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2y\right\}$

$+ \hat{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2z\right\}$.

$$\Rightarrow \nabla\phi = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} = -\frac{(3\hat{i}+\hat{j}+2\hat{k})}{14\sqrt{14}} \text{ at } P(3,1,2).$$

Let $\vec{a} = yz\hat{i} + xz\hat{j} + xy\hat{k}$, then

$$\hat{a} = \frac{yz\hat{i} + xz\hat{j} + xy\hat{k}}{\sqrt{y^2z^2+x^2z^2+x^2y^2}} = \frac{2\hat{i}+6\hat{j}+3\hat{k}}{7} \text{ at } P(3,1,2).$$

Hence directional derivative

$$\frac{d\phi}{ds} = \nabla\phi\cdot\hat{a} = -\frac{(3\hat{i}+\hat{j}+2\hat{k})}{14\sqrt{14}}\cdot\frac{(2\hat{i}+6\hat{j}+3\hat{k})}{7}$$

$$= -\frac{[(2)(3)+(1)(6)+(2)(3)]}{7\cdot 14\cdot\sqrt{14}} = -\frac{9}{49\sqrt{14}}$$

Ex6: Find the directional derivative of $\phi = xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2+4=0$ at $(2,-1,1)$. (AKTU-2011).

Soln: Given $\phi = xy^2+yz^3$

then $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z} = \hat{i}y^2+\hat{j}(2xy+z^3)+\hat{k}(3yz^2)$

$$= \hat{i}-3\hat{j}-3\hat{k} \text{ at } (2,-1,1).$$

Let $f = x\log z - y^2+4=0$.

Then $\nabla f = \hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z} = \hat{i}(\log z)+\hat{j}(-2y)+\hat{k}\left(\frac{x}{z}\right).$

$$= 0\hat{i}+2\hat{j}+2\hat{k}. \quad \text{At } (2,-1,1)$$

Then unit normal $\hat{a} = \frac{\nabla f}{|\nabla f|} = \frac{0\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{0^2+2^2+2^2}} = \frac{0\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{8}}.$

Hence directional derivative is $= \nabla\phi\cdot\hat{a}$

$$= (\hat{i}-3\hat{j}-3\hat{k})\cdot\frac{(0\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{8}}$$

$$= \frac{0-6-6}{\sqrt{8}}$$

$$= \frac{-12}{\sqrt{8}} = -3\sqrt{2}.$$

Ex-7: Find the directional derivative of scalar funcⁿ $f(x,y,z) = xyz$ at point $P(1,1,3)$ in the direction of the outward drawn normal of the surface $x^2+y^2+z^2=11$ through the point P. [Ans - $\frac{9}{\sqrt{11}}$]. (AKTU-2010)

Ex-8: Find the directional derivative of $\vec{V}^2$ where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point $(2,0,3)$ in the direction of the outward normal to the sphere $x^2+y^2+z^2 = 14$ at the point $(3,2,1)$. (AKTU-2013) (2007). [Ans - $\frac{1404}{\sqrt{14}}$].

Hint: [find $\vec{V}^2 = \vec{V}\cdot\vec{V} = (xy^2)^2 + (zy^2)^2 + (xz^2)^2$

$$= x^2y^4 + z^2y^4 + x^2z^4 = \phi \text{ (say)}.$$

Find $\nabla\phi$ at $(2,0,3)$.

Let $f = x^2+y^2+z^2-14$ find $\hat{a} = \frac{\nabla f}{|\nabla f|}$ at $(3,2,1)$]

Ex-9: Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. (AKTU-2008, 2017).

Solⁿ: Given $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$.

Then
$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$
$$= \hat{i}\left(10xy + \frac{5}{2}z^2\right) + \hat{j}(5x^2 - 10yz)\hat{j} + [-5y^2 + 5zx]\hat{k}$$
$$= \frac{25}{2}\hat{i} - 5\hat{j} + 0\hat{k} \quad \text{at } (1,1,1).$$

Given line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ [by $a_1=2, a_2=-2, a_3=1$]

the vector along this line is $\vec{a} = 2\hat{i} - 2\hat{j} + 1\hat{k}$

then
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 2\hat{j} + 1\hat{k}}{\sqrt{2^2+(-2)^2+1^2}} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k}).$$

$$\therefore D.D = \nabla\phi\cdot\hat{a} = \left(\frac{25}{2}\hat{i} - 5\hat{j} + 0\hat{k}\right)\cdot\frac{(2\hat{i} - 2\hat{j} + \hat{k})}{3}$$
$$= \frac{\left(\frac{25}{2}\right)(2) + (-5)(-2) + 0\cdot 1}{3}$$
$$= \frac{25+10}{3} = \frac{35}{3}.$$

Ex-10: Find the directional derivative of $f = e^{2x}\cos yz$ at $(0,0,0)$ in the direction of the tangent to the curve $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \frac{\pi}{4}$. [Ans -1].

Hint: [find ∇f at $(0,0,0)$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = a\sin t\,\hat{i} + a\cos t\,\hat{j} + at\,\hat{k}$

then $\hat{a} = \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \dfrac{\hat{i}}{2} - \dfrac{\hat{j}}{2} + \dfrac{\hat{k}}{\sqrt{2}}$ at $t = \frac{\pi}{4}$]

Ex-11: Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. (AKTU-2011). [Ans $\rightarrow -\frac{1}{r^2}$].

Ex-12: Find the directional derivative of $\frac{1}{r^2}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. (AKTU-2006, 2017).

Sol^n: Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow (1)$

& then $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

or $r^2 = x^2+y^2+z^2 \longrightarrow (2)$

From (2), $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$ $\Big\} \rightarrow (3)$

Now Let $\phi = \frac{1}{r^2}$.

Then $\nabla\phi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\dfrac{1}{r^2}$.

$$= \hat{i}\frac{\partial}{\partial x}\cdot\frac{1}{r^2} + \hat{j}\frac{\partial}{\partial y}\cdot\frac{1}{r^2} + \hat{k}\frac{\partial}{\partial z}\cdot\frac{1}{r^2}$$

$$= \hat{i}\frac{(-2)}{r^3}\frac{\partial r}{\partial x} + \hat{j}\frac{(-2)}{r^3}\frac{\partial r}{\partial y} + \hat{k}\frac{(-2)}{r^3}\frac{\partial r}{\partial z}.$$

$$= -\frac{2}{r^3}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] \quad \text{(from (3))}$$

$$= -\frac{2}{r^4}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{2}{r^4}\vec{r}.$$

Also $\hat{a} = \hat{r} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\vec{r}}{r}$.

Hence directional derivative of $\frac{1}{r^3}$ in the direction of $\vec{r}$

is $= \nabla\phi\cdot\hat{a} = -\dfrac{2}{r^4}\vec{r}\cdot\dfrac{\vec{r}}{r} = -\dfrac{2}{r^5}(\vec{r}\cdot\vec{r}) = -\dfrac{2}{r^5}\cdot r^2$

$= -\dfrac{2}{r^3}$.

Ex-12: Find the directional derivative of $\frac{1}{r^n}$ in the direction of $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. [Ans $-\frac{n}{r^{n+1}}$]

Ex-13: What is greatest rate of increase of $u = xyz^2$ at the point $(1,0,3)$?. [Ans - 9] (Hint find $|\nabla u|$)

Ex-14: In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum & what is its magnitude.
[Ans $\nabla\phi = 96(\hat{i} + 3\hat{j} - 3\hat{k})$ & $|\nabla\phi| = 96\sqrt{19}$].

(Angle of intersection)

The angle b/w the two surfaces = the angle b/w the normal to the surface.

$$\text{i.e.}\quad \boxed{\cos\theta = \hat{N}_1 \cdot \hat{N}_2}$$

Ex-1: Find the angle b/w the surfaces $x^2+y^2+z^2=9$ & $z = x^2+y^2-3$ at the point $(2,-1,2)$. (AKTU-2015).

Sol^n: Let $\phi_1 = x^2+y^2+z^2-9=0$.

Then
$$\nabla\phi_1 = \hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}$$
$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$
$$= 4\hat{i} - 2\hat{j} + 4\hat{k}. \qquad \text{At } (2,-1,2).$$

Then unit normal
$$\hat{N}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{4\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{4^2+(-2)^2+4^2}} = \frac{4\hat{i}-2\hat{j}+4\hat{k}}{6}$$

Now let, $\phi_2 = z - x^2 - y^2 - 3 = 0$

Then
$$\nabla\phi_2 = \hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z} = \hat{i}(-2x) + \hat{j}(-2y) + \hat{k}\cdot 1$$
$$= -4\hat{i} + 2\hat{j} + \hat{k} \qquad \text{At } (2,-1,2)$$

Then unit normal
$$\hat{N}_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{-4\hat{i}+2\hat{j}+\hat{k}}{\sqrt{(-4)^2+2^2+1^2}} = \frac{-4\hat{i}+2\hat{j}+\hat{k}}{\sqrt{21}}.$$

Hence angle b/w the two surfaces = the angle b/w the normal to the surface

$$\Rightarrow \cos\theta = \hat{N}_1\cdot\hat{N}_2 = \frac{(4\hat{i}-2\hat{j}+4\hat{k})}{6}\cdot\frac{(-4\hat{i}+2\hat{j}+\hat{k})}{\sqrt{21}}$$
$$= \frac{-16-4+4}{6\sqrt{21}} = \frac{-16}{6\sqrt{21}}$$

$$\Rightarrow \boxed{\cos\theta = \frac{-16}{6\sqrt{21}}} = \boxed{\frac{-8}{3\sqrt{21}}}$$

Ex-2→ Calculate the angle b/w the normals to the surface $xy = z^2$ at the points $(4,1,2)$ & $(3,3,-3)$. (AKTU-2012).

[Hint find $\hat{N}_1 = \frac{\nabla\phi}{|\nabla\phi|}$ at $(4,1,2)$ & $\hat{N}_2 = \frac{\nabla\phi}{|\nabla\phi|}$ at $(3,3,-3)$.] [Ans - $\cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$.

Ex-3→ If θ is the acute angle b/w the surface $xy^2z = 3x + z^2$ & $3x^2 - y^2 + 2z = 1$ at the point $(1,-2,1)$ show that

$$\cos\theta = \frac{3}{7\sqrt{6}}.$$

Tangent Plane and Normal

① The tangent plane to the surface $f(x,y,z)$ at point (x_0,y_0,z_0) is given by

$$(x-x_0)\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0,z_0)} + (y-y_0)\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0,z_0)} + (z-z_0)\left(\frac{\partial f}{\partial z}\right)_{(x_0,y_0,z_0)} = 0.$$

② Also the eqⁿ of normal to the surface at point (x_0,y_0,z_0) is

$$\frac{(x-x_0)}{\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0,z_0)}} = \frac{(y-y_0)}{\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0,z_0)}} = \frac{(z-z_0)}{\left(\frac{\partial f}{\partial z}\right)_{(x_0,y_0,z_0)}}.$$

Ex-1→ find the tangent plane and normal line to the surface $x^2+y^2+z^2 = 30$ at $(1,-2,5)$.

Solⁿ→ Let $f(x,y,z) = x^2+y^2+z^2-30 = 0$

Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 2z$

At $(1,-2,5)$ $\frac{\partial f}{\partial x} = 2$, $\frac{\partial f}{\partial y} = -4$, $\frac{\partial f}{\partial z} = 10$.

∴ The eqⁿ of tangent plane is

$$(x-x_0)\frac{\partial f}{\partial x} + (y-y_0)\frac{\partial f}{\partial y} + (z-z_0)\frac{\partial f}{\partial z} = 0$$

$$\Rightarrow (x-1)\cdot 2 + (y+2)\cdot(-4) + (z-5)\cdot 10 = 0$$

$$\Rightarrow \boxed{x - 2y + 5z = 30}$$

Also eqⁿ of normal is

$$\frac{x-x_0}{\frac{\partial f}{\partial x}} = \frac{y-y_0}{\frac{\partial f}{\partial y}} = \frac{z-z_0}{\frac{\partial f}{\partial z}} \Rightarrow \boxed{\frac{x-1}{2} = \frac{y+2}{-4} = \frac{z-5}{10}}$$

Ex: find the directional derivative of $\nabla\cdot(\nabla f)$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x+z^2$ where $f = 2x^3y^2z^4$ (AKTU-2008, 2013)

Solⁿ:

MATHEMATICS-I
KAS-103T
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Module-V
* Vector Calculus *
[redacted]

Divergence of a vector point func^n→

The divergence of a differentiable vector point func^n $\vec{V}$ is denoted by div$\vec{V}$ and is defined by

$$\boxed{\begin{aligned}\text{div}\vec{V} = \nabla\cdot\vec{V} &= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{V}\\ &= \hat{i}\cdot\frac{\partial\vec{V}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{V}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{V}}{\partial z}.\end{aligned}}$$

Note→ ① If $\vec{V}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$ then

$$\text{div}\vec{V} = \nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(f_1\hat{i}+f_2\hat{j}+f_3\hat{k})$$

$$= \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

$[\because \hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$
$\& \ \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0]$

② Divergence of a vector point function is a scalar point func^n.

Imp→ **Solenoidal Vector→** If div $\vec{V}=0$, then $\vec{V}$ is called solenoidal vector point func^n.

Curl of a vector point func^n→

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\boxed{\begin{aligned}\text{curl}\,\vec{V} = \nabla\times\vec{V} &= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{V}\\ &= \hat{i}\times\frac{\partial\vec{V}}{\partial x}+\hat{j}\times\frac{\partial\vec{V}}{\partial y}+\hat{k}\times\frac{\partial\vec{V}}{\partial z}\end{aligned}}$$

Note→ ① If $\vec{V}=f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$ then curl $\vec{V}=\nabla\times\vec{V}$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$$

② Curl of a vector point func^n is a vector point func^n.

Imp→ Irrotational Vector→ If curl $\vec{V} = 0$, then $\vec{V}$ is called irrotational vector.

Note→ Diverengence & curl of a constant vector $\vec{a}$ are 0.

i.e $\text{div}\,\vec{a} = 0$ & $\text{curl}\,\vec{a} = \vec{0}$.

Ex-1→ If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that

(i) $\text{div}\,\vec{r} = 3$ (ii) $\text{curl}\,\vec{r} = 0$. (AKTU-2014, 2018).

Solⁿ→ Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

(i) $\text{div}\,\vec{r} = \nabla\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3.$$

(ii) $\text{curl}\,\vec{r} = \nabla\times\vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}.$$

$\Rightarrow$ $\text{curl}\,\vec{r} = \vec{0}$. Hence $\vec{r}$ is a irrotational vector.

Ex-2→ Show that $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ is salenoidal.

Ex-3→ Prove that $\vec{V} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is salenoidal.

Ex-4→ Find the value of 'p' if the vector $\vec{V} = (x+py)\hat{i} + (py-3z)\hat{j} + (x-2z)\hat{k}$ is salenoidal. (AKTU-2008).

Solⁿ→ If $\vec{V}$ is salenoidal vector then $\text{div}\,\vec{V} = 0$

$$\Rightarrow \nabla\cdot\vec{V} = 0 \Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(x+py)\hat{i} + (yp-3z)\hat{j} + (x-2z)\hat{k}\right] = 0.$$

$\Rightarrow \frac{\partial}{\partial x}(x+py) + \frac{\partial}{\partial y}(yp-3z) + \frac{\partial}{\partial z}(x-2z) = 0$

$\Rightarrow 1 + p - 2 = 0 \Rightarrow p-1=0 \Rightarrow \boxed{p=1}$.

Ex → Find m, if $\vec{F} = mx\hat{i} - 5y\hat{j} + 2z\hat{k}$ is a solenoidal vector. (AKTU-2018) [Am-3]

Ex-5 → Find the divergence and curl of the vector $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point $(2,-1,1)$.

Ex-6 → Find div $\vec{f}$ and curl $\vec{f}$ of the vector

$\vec{f} = (x^2+yz)\hat{i} + (y^2+zx)\hat{j} + (z^2+xy)\hat{k}$ at $(1,1,1)$. (AKTU-2015).

Solⁿ → Given

$$\vec{f} = (x^2+yz)\hat{i} + (y^2+zx)\hat{j} + (z^2+xy)\hat{k}$$

Now div $\vec{f} = \nabla \cdot \vec{f}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{f}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(x^2+yz)\hat{i} + (y^2+zx)\hat{j} + (z^2+xy)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2+yz) + \frac{\partial}{\partial y}(y^2+zx) + \frac{\partial}{\partial z}(z^2+xy).$$

$$= 2x + 2y + 2z$$

$$= 6 \quad [\text{at } (1,1,1)].$$

Also curl $\vec{f} = \nabla \times \vec{f}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+yz & y^2+zx & z^2+xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2+xy) - \frac{\partial}{\partial z}(y^2+zx)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2+xy) - \frac{\partial}{\partial z}(x^2+yz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2+zx) - \frac{\partial}{\partial y}(x^2+yz)\right].$$

$$= \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\Rightarrow$ curl $\vec{f} = \vec{0}$. Hence $\vec{f}$ is a irrotational vector.

$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$

Ex-7 Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$.

[Ans div $\vec{F} = 6(x+y+z)$ & curl $\vec{F} = \vec{0}$].

Ex-8 Find div & curl for the functions

(i) $\vec{F}(x,y,z) = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

(ii) $\vec{V} = x^2y^2\hat{i} + 2xy\hat{j} + (y^2-xy)\hat{k}$ at $(1,2,3)$.

Ex-8 If $\vec{V} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ then show that

$$\text{div}\,\vec{V} = \frac{2}{\sqrt{x^2+y^2+z^2}} \quad \& \text{ curl } \vec{V} = 0. \quad \text{(AKTU-2011)}.$$

Velocity Potential or Scalar Potential

Note → If $\vec{V}$ is irrotational then

$$\text{curl}\,\vec{V} = 0$$

Then, let ϕ is scalar potential. To find ϕ, let

$$\vec{V} = \nabla\phi$$

Then $\vec{V}\cdot\vec{dr} = \nabla\phi\cdot\vec{dr}$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi \qquad \text{[By total differentiation]}$$

$$\Rightarrow d\phi = \vec{V}\cdot\vec{dr}$$

$$\Rightarrow \boxed{\phi = \int\vec{V}\cdot\vec{dr}}$$

Ex-1 A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

Is this motion irrotational? If so, find the velocity potential.

(AKTU-2016).

Sol^n Given $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

Now
$$\text{curl}\,\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow$ curl $\vec{V} = \vec{0}$. Hence $\vec{V}$ is irrotational vector.
or motion is irrotational.

Now, let $\vec{V} = \text{grad}\,\phi$, where ϕ is velocity potential.

$$\Rightarrow \vec{V}\cdot\vec{dr} = \nabla\phi\cdot\vec{dr}$$

$$\Rightarrow \vec{V}\cdot\vec{dr} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi$$

$$\Rightarrow d\phi = \vec{V}\cdot\vec{dr}$$

$$\Rightarrow d\phi = [(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}]\cdot[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$$

$$= (y+z)dx + (z+x)dy + (x+y)dz$$

$$= (y\,dx + x\,dy) + (z\,dx + x\,dz) + (y\,dz + z\,dy)$$

$$\Rightarrow d\phi = d(xy) + d(xz) + d(yz)$$

On integrating, $\boxed{\phi = xy + xz + yz + C}$

Ex-2: Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational, find the velocity potential ϕ such that $\vec{A} = \nabla\phi$.
(AKTU-2011, 2014).

Sol^n: Given $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$.

Prove curl $\vec{A} = 0$ (try yourself).

Now, let $\vec{A} = \nabla\phi$.

then $\vec{A}\cdot\vec{dr} = \nabla\phi\cdot\vec{dr} = \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = d\phi$

$$\Rightarrow d\phi = \vec{A}\cdot\vec{dr}$$

$$= [(6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}]\cdot[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$$

$$= (6xy + z^3)dx + (3x^2 - z)dy + (3xz^2 - y)dz$$

$$= (6xy\,dx + 3x^2dy) + (z^3dx + 3xz^2dz) - (z\,dy + y\,dz)$$

$$\Rightarrow d\phi = d(3x^2y) + d(z^3x) - d(yz)$$

On integration, $\boxed{\phi = 3x^2y + z^3x - yz + C}$

Ex-3: Find the constants a, b, c so that
$$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$ is
irrotational. If $\vec{F} = \text{grad}\,\phi$, show that
$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz.$$ (AKTU-2011, 2014, 2018).

Solⁿ: Given $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$

Since $\vec{F}$ is irrotational then

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right] + \hat{k}\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right] = \vec{0}$$

$$\Rightarrow \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow c+1=0, \quad a-4=0, \quad b-2=0$$

$$\Rightarrow c=-1, \quad a=4, \quad b=2.$$

Hence $\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x+(-1)y+2z)\hat{k}$.

Let $\vec{F} = \text{grad}\,\phi$

then $\vec{F}\cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi.$$

$$\Rightarrow d\phi = \vec{F}\cdot d\vec{r}$$
$$= (x+2y+4z)\,dx + (2x-3y-z)\,dy + (4x-y+2z)\,dz$$
$$= x\,dx + (2y\,dx + 2x\,dy) + (4z\,dx + 4x\,dz) - 3y\,dy - (z\,dy + y\,dz) + 2z\,dz$$

$$\Rightarrow d\phi = x\,dx + 2d(xy) + 4d(xz) - 3y\,dy - d(yz) + 2z\,dz$$

On integrating,
$$\boxed{\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2 + C}$$

Hence proved.

Ex-4: Determine the constants a & b such that the curl of vector
$\vec{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero.
i.e $\vec{A}$ is irrotational. (AKTU-2008). [Ans $a = -3, b = 8$,

Ex-5: If a vector field is given by $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$.
Is this field irrotational. If so, find its scalar potential. (AKTU-2014, 2009) [yes, $\phi = \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$.

Ex-6: A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$
Show that the field is irrotational and find the scalar potential. (AKTU-2002, 2003). [Ans $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + c$].

Ex-7: A fluid motion is given by
$$\vec{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}.$$
Is the motion irrotational? If so, find the velocity potential.
(AKTU - 2004, 2006, 2011).

[Hint $d\phi = (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (xy\cos z + y^2)\,dz$

$$= (y\sin z\,dx + x\sin z\,dy + xy\cos z\,dz) - \sin x\,dx + (2yz\,dy + y^2\,dz)$$

$\Rightarrow d\phi = d(xy\sin z) - \sin x\,dx + d(y^2 z)$

on integr $\phi = xy\sin z + \cos x + y^2 z + c$.

Ex-8: Prove that
$$\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$$
is both solenoidal and irrotational. Find scalar potential ϕ.

[Ans $\phi = xy^2 - xz^2 + z^2 - x^2 + 3xyz + c$]. (AKTU - 2012).

Ex-9: If $\vec{A} = xz^2\hat{i} + 2y\hat{j} - 3xz\hat{k}$
& $\vec{B} = 3xz\hat{i} + 2yz\hat{j} - z^2\hat{k}$

Find the value of $\vec{A} \times (\nabla \times \vec{B})$ & $(\vec{A} \times \nabla) \times \vec{B}$.
(AKTU-2017)

Ans→ $9x^2z\hat{i} + 6xyz\hat{j} + (3x^2z^2 + 4y^2)\hat{k}$
$-2yz\hat{i} + (4y^2 + 15xz^2 + 3x^2z^2)\hat{k}$.

Ex → (1) Prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$. Hence evaluate

$\nabla^2(\log r)$ if $r = (x^2+y^2+z^2)^{1/2}$. (AKTU - 2010, 2012).

Sol^n We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \rightarrow (1)$

& $r^2 = x^2+y^2+z^2 \longrightarrow (2)$

From (2), $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$ $\Big\} \rightarrow (3)$

Now $\nabla^2 f(r) = (\nabla \cdot \nabla) f(r)$

$$= \left[\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\right] f(r)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r).$$

$$\Rightarrow \nabla^2 f(r) = \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r) \longrightarrow (4)$$

Now $\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right]$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{r}\right]$$

$$= f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial x} f'(r)\cdot\frac{x}{r}$$

$$= f'(r)\left[\frac{1}{r} + \frac{(-x)}{r^2}\frac{\partial r}{\partial x}\right] + f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r}$$

$$= f'(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right] + \frac{x^2}{r^2} f''(r) \quad [\text{from } (3)]$$

$$= \frac{f'(r)}{r} - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \longrightarrow (5)$$

Similarly, $\frac{\partial^2}{\partial y^2} f(r) = \frac{f'(r)}{r} - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r) \longrightarrow (6)$

& $\frac{\partial^2}{\partial z^2} f(r) = \frac{f'(r)}{r} - \frac{z^2}{r^3} f'(r) + \frac{z^2}{r^2} f''(r) \longrightarrow (7)$

Using in (4),

$$\nabla^2 f(r) = \frac{3f'(r)}{r} - \frac{f'(r)}{r^3}(x^2+y^2+z^2) + \frac{(x^2+y^2+z^2)}{r^2} f''(r)$$

$$\Rightarrow \boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$$

Put $f(r) = \log r$, we get $\nabla^2 \log r = -\frac{1}{r^2} + \frac{2}{r}\cdot\frac{1}{r} = \frac{1}{r^2} = \frac{1}{x^2+y^2+z^2}$.

Vector identities: Prove that

① $\text{curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi = \vec{0}$. (AKTU-2011, 2018).

Proof: we have
$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Now $\text{curl}(\text{grad}\,\phi) = \nabla \times \nabla\phi$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} \qquad \left[\because \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$\Rightarrow \boxed{\text{curl}(\text{grad}\,\phi) = 0}$$

② Prove that $\text{div}(\text{curl}\,\vec{V}) = \nabla\cdot(\nabla\times\vec{V}) = 0$ (AKTU - 2011, 2016, 2018)

Proof: Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ then

$\text{curl}\,\vec{V}$ or $\nabla\times\vec{V}$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right).$$

Hence $\text{div}(\text{curl}\,\vec{V}) = \nabla\cdot(\nabla\times\vec{V})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x\partial y} - \frac{\partial^2 V_2}{\partial x\partial z} - \frac{\partial^2 V_3}{\partial y\partial x} + \frac{\partial^2 V_1}{\partial y\partial z} + \frac{\partial^2 V_2}{\partial z\partial x} - \frac{\partial^2 V_1}{\partial z\partial y}$$

$$= 0. \quad \Rightarrow \boxed{\text{div}(\text{curl}\,\vec{V}) = 0}$$

(3) Prove that $\text{div}(\text{grad}\,\phi) = \nabla\phi$

Proof: $\text{div}(\text{grad}\,\phi) = \nabla\cdot(\nabla\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi.$$

(4) $\boxed{\text{Curl}(\text{Curl}\,\vec{V}) = \text{grad}(\text{div}\,\vec{V}) - \nabla^2\vec{V}}$

(5) If $\vec{A}$ is a vector func^n and ϕ is a scalar func^n

v.imp then $\text{div}(\phi\vec{A}) = \phi\,\text{div}\,\vec{A} + (\text{grad}\,\phi)\cdot\vec{A}$

or $\nabla\cdot(\phi\vec{A}) = \phi(\nabla\cdot\vec{A}) + (\nabla\phi)\cdot\vec{A}$.

Ex→(1) Prove that $\text{div}(\text{grad}\,r^n) = \nabla^2(r^n) = n(n+1)r^{n-2}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$.

(AKTU-2012, 2014).

Sol^n: We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow (1)$

& $r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2 \longrightarrow (2)$

From (2), $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$ $\Bigg\} \longrightarrow (3)$

Now $\text{grad}\,r^n = n r^{n-1}\hat{r} \longrightarrow (4)$

$$= n r^{n-1}\frac{\vec{r}}{r}$$

$\Rightarrow \text{grad}\,r^n = n r^{n-2}\vec{r} \longrightarrow (5)$

$\therefore \text{div}(\text{grad}\,r^n) = \text{div}(n r^{n-2}\vec{r})$

$$= (n r^{n-2})\,\text{div}\,\vec{r} + \text{grad}(n r^{n-2})\cdot\vec{r}$$

$[\because \text{div}(\phi\vec{A}) = \phi\,\text{div}\,\vec{A} + (\text{grad}\,\phi)\cdot\vec{A}.$

$$= n r^{n-2}\cdot 3 + [n(n-2)r^{n-3}\hat{r}]\cdot\vec{r}$$

[from (4)].

$$\Rightarrow \text{div}(\text{grad}\, r^n) = 3n r^{n-2} + n(n-2) r^{n-3}\left(\frac{\vec{r}}{r}\cdot\vec{r}\right) \quad \left[\because \hat{r} = \frac{\vec{r}}{r}\right]$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= [3n + n^2 - 2n]\, r^{n-2}$$

$$= (n^2 + n)\, r^{n-2}$$

$$\Rightarrow \text{div}(\text{grad}\, r^n) = n(n+1)\, r^{n-2}$$

$$\Rightarrow \boxed{\nabla^2(r^n) = n(n+1)\, r^{n-2}}$$

Put $n = -1$, we get

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = (-1)(-1+1)\, r^{-3} = 0}$$

Ex-21 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, show that $\text{div}\, \hat{r} = \frac{2}{r}$.

Solⁿ: By the vector identity,

$$\text{div}(\phi\vec{A}) = \phi\, \text{div}\, \vec{A} + (\nabla\phi)\cdot\vec{A} \longrightarrow (1)$$

Now ~~div r~~ $\text{div}\, \hat{r} = \text{div}\left(\frac{\vec{r}}{r}\right)$

$$= \text{div}\left(\frac{1}{r}\vec{r}\right)$$

$$= \frac{1}{r}\text{div}\, \vec{r} + \left(\nabla\frac{1}{r}\right)\cdot\vec{r}$$

$$= \frac{1}{r}\times 3 + (-1) r^{-2}\, \hat{r}\cdot\vec{r} \qquad \left[\nabla\frac{1}{r} = (-1) r^{-2}\, \hat{r}\right]$$

$$= \frac{3}{r} - \frac{1}{r^2}\times\frac{\vec{r}}{r}\cdot\vec{r}$$

$$= \frac{3}{r} - \frac{1}{r^2}\times\frac{r^2}{r} \qquad \left[\because \hat{r} = \frac{\vec{r}}{r}\ \&\ \vec{r}\cdot\vec{r} = r^2\right]$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

(6) If $\vec{A}$ is a vector function and ϕ is a scalar function then

$$\text{Curl}(\phi\vec{A}) = \phi\, \text{curl}\, \vec{A} + (\text{grad}\, \phi)\times\vec{A}.$$

Ex-1: Prove that $\text{curl}(r^n \vec{r}) = \vec{0}$.

Sol^n: By vector identity, we have

$$\text{curl}(\phi \vec{A}) = \phi \,\text{curl}\,\vec{A} + (\text{grad}\,\phi) \times \vec{A} \longrightarrow (1)$$

Now $\text{curl}(r^n \vec{r}) = r^n \,\text{curl}\,\vec{r} + \text{grad}\, r^n \times \vec{r}$

$$= r^n \cdot \vec{0} + n r^{n-1} \hat{r} \times \vec{r}$$

$$= \vec{0} + n r^{n-1} \frac{\vec{r}}{r} \times \vec{r} \quad \left[\because \hat{r} = \frac{\vec{r}}{r}\right]$$

$$= \vec{0} \quad [\vec{r} \times \vec{r} = \vec{0}].$$

$\Rightarrow \text{curl}(r^n \vec{r}) = \vec{0}$.

Ex-2: Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. Find the scalar potential. (AKTU-2018, 2007).

Sol^n:

* By vector identity, we have

$$\text{div}(\phi \vec{A}) = \phi \,\text{div}\,\vec{A} + \text{grad}\,\phi \cdot \vec{A} \rightarrow (1)$$

Then $\text{div}\,\vec{F} = \text{div}\left(\frac{1}{r^3}\vec{r}\right) = \frac{1}{r^3}\text{div}\,\vec{r} + \text{grad}\,\frac{1}{r^3} \cdot \vec{r}$ (from (1))

$$= \frac{1}{r^3} \times 3 + \text{grad}\, r^{-3} \cdot \vec{r}$$

$$= \frac{3}{r^3} + (-3) r^{-4} \hat{r} \cdot \vec{r}$$

$$= \frac{3}{r^3} - \frac{3}{r^4} \frac{\vec{r}}{r} \cdot \vec{r} \quad \left[\because \hat{r} = \frac{\vec{r}}{r}\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^4} \times \frac{r^2}{r} \quad [\because \vec{r} \cdot \vec{r} = r^2]$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

$\Rightarrow \text{div}\,\vec{F} = 0$. Hence $\vec{F}$ is solenoidal.

* By vector identity, we have

$$\text{curl}(\phi \vec{A}) = \phi \,\text{curl}\,\vec{A} + \text{grad}\,\phi \times \vec{A} \longrightarrow (2)$$

Then $\text{curl}\,\vec{F} = \text{curl}\left(\frac{1}{r^3}\vec{r}\right) = \frac{1}{r^3}\text{curl}\,\vec{r} + \text{grad}\, r^{-3} \times \vec{r}$

$$= \frac{1}{r^3} \times \vec{0} + (-3) r^{-4} \hat{r} \times \vec{r}$$

$$= -\frac{3}{r^{-4}} \cdot \frac{\vec{r}}{r} \times \vec{r} \quad \left[\because \hat{r} = \frac{\vec{r}}{r}\right]$$

$$= \vec{0} \quad [\because \vec{r} \times \vec{r} = \vec{0}].$$

$\Rightarrow$ Curl $\vec{F} = \vec{0}$

$\vec{F}$ is irrotational.

Hence vector $\vec{F}$ is a scalar potential.

Now, let $\vec{F} = \nabla\phi$, where ϕ is a scalar potential.

$$\therefore \vec{F}\cdot d\vec{r} = \nabla\phi\cdot d\vec{r}$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi$$

$$\Rightarrow d\phi = \vec{F}\cdot d\vec{r}$$

$$= \frac{\vec{r}}{r^3}\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}}\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\Rightarrow d\phi = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow d\phi = d\left[-(x^2+y^2+z^2)^{-1/2}\right]$$

On integrating, $\phi = -(x^2+y^2+z^2)^{-1/2} + C$.

$$\Rightarrow \boxed{\phi = -\frac{1}{r} + C}$$

Ex→2) Prove that $\operatorname{div}(r^n\vec{r}) = (n+3)r^n$. further show that $r^n\vec{r}$ is solenoidal ~~only~~ only if $n = -3$.

Ex-3) If $u = x^2+y^2+z^2$ & $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$

show that $\operatorname{div}(u\vec{V}) = 54$.

Ex-4) Show that $\nabla^2\left(\frac{x}{r^3}\right) = 0$ where r is magnitude of position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

MATHEMATICS-I
KAS-103T
* Lecture No - 45 *

Module - V
Vector Calculus

[redacted]

Line Integral:-

Let $\vec{F}$ represent the force acting on a particle moving along arc C. The work done

$$\boxed{W = \int_C \vec{F} \cdot \vec{dr}}$$

where $\int_C \vec{F} \cdot \vec{dr}$ is called line integral

If C is any closed curve, then work done

$$\boxed{W = \oint_C \vec{F} \cdot \vec{dr}}$$

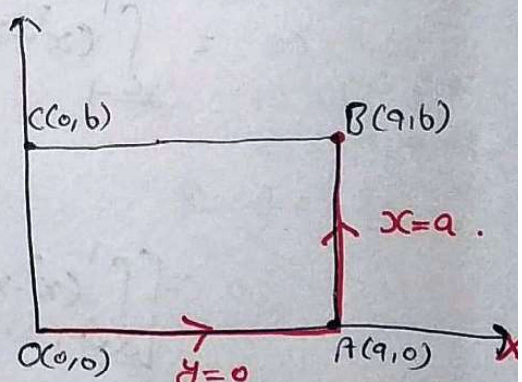
Note: If $\oint_C \vec{F} \cdot \vec{dr} = 0$, then $\vec{F}$ is said to to irrotational.

Ex:- Find the total work done by the force $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ in moving a point from (0,0) to (a,b) along the rectangle bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$. (AKTU-2011, 2014).

Sol: Given $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$

Since $\vec{r} = x\hat{i} + y\hat{j}$ then $\vec{dr} = dx\hat{i} + dy\hat{j}$

$\therefore \vec{F} \cdot \vec{dr} = (x^2+y^2)dx - 2xy\,dy \longrightarrow (1)$



Now Total work done

$$W = \int_C \vec{F} \cdot \vec{dr}$$

$$= \int_{OA} \vec{F} \cdot \vec{dr} + \int_{AB} \vec{F} \cdot \vec{dr}$$

for OA,
$y=0$, $dy=0$
& $x=0$ to $x=a$.

Also for, AB
$x=a$, $dx=0$
& $y=0$ to $y=b$.

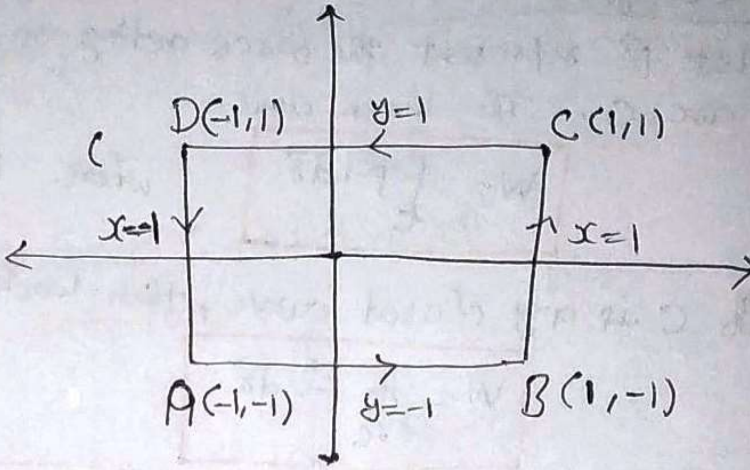
$$= \int_{x=0}^{a} x^2 dx + \int_{y=0}^{b} (-2ay)dy$$

$$= \left[\frac{x^3}{3}\right]_0^a + (-2a)\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{a^3}{3} - ab^2 ..$$

Ex-2: Evaluate the line integrals $\int_C [(x^2+xy)\,dx + (x^2+y^2)\,dy]$ where C is the square formed by the lines $y=\pm 1$ and $x=\pm 1$.
(AKTU-2007, 2013).

Solⁿ:



* for AB, $y=-1$ & $dy=0$
  & $x=-1$ to $x=1$
* for BC, $x=1$ & $dx=0$
  & $y=-1$ to $y=1$
* for CD, $y=1$, $dy=0$
  & $x=1$ to $x=-1$
* for DA, $x=-1$ & $dx=0$
  & $y=1$ to $y=-1$.

Hence $\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$

$$= \int_{x=-1}^{1} (x^2-x)\,dx + \int_{y=-1}^{1} (1+y^2)\,dy + \int_{x=1}^{-1} (x^2+x)\,dx + \int_{y=1}^{-1} (1+y^2)\,dy$$

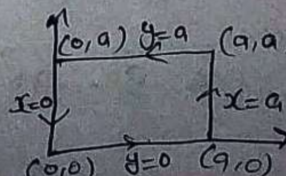
$$= \left[\int_{x=-1}^{1} (x^2-x)\,dx - \int_{x=-1}^{1} (x^2+x)\,dx\right] + \cancel{\int_{y=-1}^{1} (1+y^2)\,dy - \int_{y=-1}^{1} (1+y^2)\,dy}$$

$$= \int_{x=-1}^{1} (-2x)\,dx = -2\left[\frac{x^2}{2}\right]_{-1}^{1}$$

$$= -\left[1^2-(-1)^2\right] = -[1-1] = 0.$$

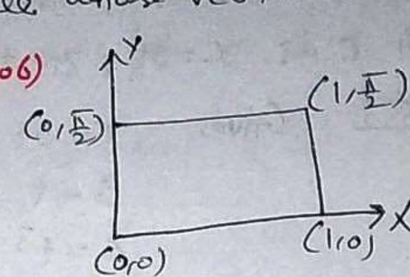
Ex-3: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of a square in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ & $y=a$. (AKTU-2015).

Ans $\frac{a^3}{2}$.

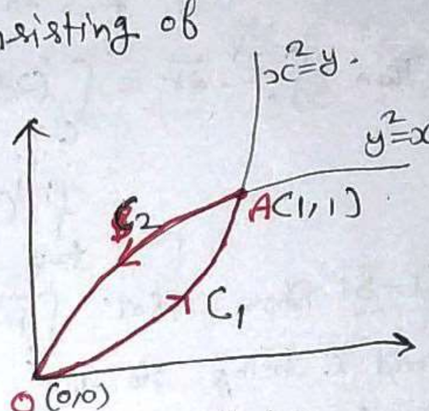


Ex-4 → Find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = e^x \sin y\,\hat{i} + e^x \cos y\,\hat{j}$ & C is the rectangle whose vertices are $(0,0)$, $(1,0)$, $(1,\frac{\pi}{2})$, $(0,\frac{\pi}{2})$. (AKTU-2006)

[Ans – 0].



Ex-5 → If $\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$, evaluate $\oint_C \vec{A}\cdot\vec{dr}$ around the curve C consisting of $y=x^2$ and $y^2=x$. (AKTU – 2014, 2018).



Sol^n → Given

$\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$

Since $\vec{r} = x\hat{i} + y\hat{j}$ then $\vec{dr} = dx\,\hat{i} + dy\,\hat{j}$

$\therefore\ \vec{A}\cdot\vec{dr} = (x-y)\,dx + (x+y)\,dy \longrightarrow (1)$

[On solving $y^2=x$ & $x^2=y$ we get $x^4=x$

$\Rightarrow x^4-x=0$

$\Rightarrow x(x^3-1)=0$

$\Rightarrow x=0,\ x=1$

then $y=0,\ y=1$]

⊛ For C_1, $y=x^2$ & $dy = 2x\,dx$
& $x=0$ to $x=1$.

⊛ For C_2, $x=y^2$ & $dx = 2y\,dy$
& $y=1$ to $y=0$.

Now

$$W = \oint_C \vec{A}\cdot\vec{dr} = \int_{C_1} \vec{A}\cdot\vec{dr} + \int_{C_2} \vec{A}\cdot\vec{dr}$$

$$= \int_{x=0}^{1} [(x-x^2)\,dx + (x+x^2)\cdot 2x\,dx] + \int_{y=1}^{0} [(y^2-y)\,2y\,dy + (y^2+y)\,dy]$$

$$= \int_{x=0}^{1} [x + x^2 + 2x^3]\,dx + \int_{y=1}^{0} [2y^3 - y^2 + y]\,dy.$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{2}\right]_0^1 + \left[\frac{y^4}{2} - \frac{y^3}{3} + \frac{y^2}{2}\right]_1^0$$

$$= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - 0\right) + \left(0 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\right).$$

$$= 1 + \frac{1}{3} - 1 + \frac{1}{3} = \frac{2}{3}.$$

Ex-6 → If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the xy-plane from $(0,0)$ to $(1,4)$ along a curve $y=4x^2$. Find work done.

Ans – $\frac{104}{5}$.

Ex-7: A vector field is given by

$\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the path C is $x=2t$, $y=t$, $z=t^3$ from $t=0$ to $t=1$.

[Hint: Given $x = 2t$ then $dx = 2t\,dt$

$y = t$ then $dy = dt$

$z = t^3$ then $dz = 3t^2 dt$

& $t=0$ to $t=1$.

Then $\int_C \vec{F}\cdot d\vec{r} = \int_C (2y+3)\,dx + xz\,dy + (yz-x)\,dz$

$= \int_{t=0}^{1} (4t+6+2t^4+3t^6-6t^3)\,dt = 7.32857.$].

Ex-8: Show that $\int_C \vec{F}\cdot d\vec{r} = 3\pi$, given that $\vec{F} = z\hat{i} + x\hat{j} + z\hat{k}$ and C being the arc of the curve $\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ from $t=0$ to $t=2\pi$. [Hint – $x = \cos t$, $y = \sin t$, $z = t$]. [Ans – 3π].

Ex-9: If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$ evaluate $\int_C \vec{F}\times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t=0$ to $t=\frac{\pi}{2}$. (AKTU – 2002).

Sol^n: We have $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$ & $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\therefore \vec{F}\times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = (-z\,dz - x\,dy)\hat{i} - (2y\,dz - x\,dx)\hat{j} + (2y\,dy + z\,dx)\hat{k}.$$

$$= [-2\cos t(-2\sin t)\,dt - \cos t(\cos t)\,dt]\,\hat{i}$$
$$- [2\sin t(-2\sin t)\,dt - \cos t(-\sin t)\,dt]\,\hat{j}$$
$$+ [2\sin t(\cos t)\,dt + 2\cos t(-\sin t)\,dt]\,\hat{k}.$$

$$= [(4\cos t\sin t - \cos^2 t)\,\hat{i} + (4\sin^2 t - \cos t\sin t)\,\hat{j}]\,dt.$$

Hence
$$\int_C \vec{F}\times d\vec{r} = \int_{t=0}^{\pi/2} [(4\cos t\sin t - \cos^2 t)\hat{i} + (4\sin^2 t - \cos t\sin t)\hat{j}]\,dt.$$

$$= \int_{t=0}^{\pi/2} \left[\left\{2\sin 2t - \frac{(\cos 2t + 1)}{2}\right\}\hat{i}\right]dt + \int_{t=0}^{\pi/2}\left[2(1-\cos 2t) - \frac{\sin 2t}{2}\right]\hat{j}\,dt$$

$$= \left[-\cos 2t - \frac{1}{4}\sin 2t - \frac{t}{2}\right]_0^{\pi/2} \hat{i} + \left[2t - \sin 2t + \frac{\cos 2t}{4}\right]_0^{\pi/2} \hat{j}.$$

$$= \left[-\cos\pi - \frac{1}{4}\sin\pi - \frac{1}{2}\frac{\pi}{2} + \cos 0 + \frac{1}{4}\sin 0 + \frac{1}{2}0\right]\hat{i}$$

$$+ \left[2\cdot\frac{\pi}{2} - \sin\pi + \frac{1}{4}\cos\pi - 0 + \sin 0 - \frac{1}{4}\cos 0\right]\hat{j}$$

$$= \left(2 - \frac{\pi}{4}\right)\hat{i} + \left(\pi - \frac{1}{2}\right)\hat{j}.$$

Ex-10 → If $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\oint \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve C $x = t,\ y = t^2,\ z = t^3$. [AKTU - 2006].

[Ans - 5]

Hint → $\left[\begin{array}{l} \text{at } (0,0,0),\ t = 0 \\ \text{at } (1,1,1),\ t = 1 \end{array}\right]$

$\Rightarrow t = 0$ to $t = 1$.

Ex-11 → Compute the work done by the force $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$, when it moves a particle from the the point $(0,0,0)$ to the point $(2,1,1)$ along the curve $x = 2t^2,\ y = t,\ z = t^3$ (AKTU-2011). [Ans - $8\frac{8}{35}$].

Hint → $\left[\begin{array}{l} \text{at } (0,0,0) \text{ then } t = 0 \\ \text{at } (2,1,1) \text{ then } t = 1 \end{array}\right. \Rightarrow t = 0 \text{ to } t = 1 \Big]$

Ex-12 (i) Suppose $\vec{F}(x,y,z) = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field, Find the work done by $\vec{F}$ along the line from $(1,2,3)$ to $(3,5,7)$. (AKTU-2005)

Sol → Eqn of line joining the points $(1,2,3)$ & $(3,5,7)$.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\Rightarrow \frac{x-1}{3-1} = \frac{y-2}{5-2} = \frac{z-3}{7-3} \Rightarrow \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = t \text{ (say)}$$

$$\Rightarrow x = 2t+1,\ y = 3t+2,\ z = 4t+3.$$

At $(1,2,3)$, $t = 0$ & At $(3,5,7)$ $t = 1$.

$$W = \oint_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} (16t^3 + 24t^2 + 37t + 20)\,dt = \frac{101}{2}.$$

(ii) Also find the work done, where C is the straight line joining $(0,0,0)$ & $(1,1,1)$.

Green's Theorem in the plane →

If C is a regular closed curve in the xy-plane and R be region bounded by C, Then

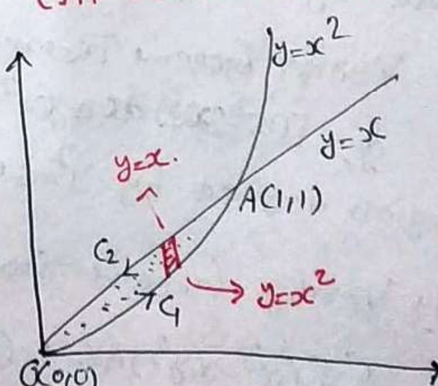
$$\boxed{\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.}$$

where $M(x,y)$ & $N(x,y)$ are continuously differentiable functions inside and on C. (AKTU-2012, 2008).

Imp Note → ① If Verify is asked in the question then solve left side and right side of green's theorem.
② If Find, Evaluate is asked in the question, then use right hand side (i.e solve by double integral).

Verify Questions

Ex-1 → Verify Green's theorem in the plane for $\int_C (xy+y^2)dx + x^2 dy$ where C is the ~~square~~ closed curve of the region bounded by $y=x$ & $y=x^2$. (AKTU-2008, 2010).



Sol → Given

$$I = \int_C (xy+y^2)\,dx + x^2\,dy \longrightarrow ①$$

Here $M = xy + y^2$, $N = x^2$

then $\frac{\partial M}{\partial y} = x + 2y$, $\frac{\partial N}{\partial x} = 2x$.

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y = x - 2y.$$

By Green's Theorem, we have $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$.

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\,dy\,dx = \int_{x=0}^{1} \left[xy - y^2\right]_{x^2}^{x} dx.$

$$= \int_{x=0}^{1} (-x^3 + x^4)\,dx = \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = -\frac{1}{20} \longrightarrow ②.$$

Also, $\oint_C M dx + N dy = \int_{C_1} M dx + N dy + \int_{C_2} M dx + N dy$.

$$= \int_{C_1} [(xy+y^2)\,dx + x^2 dy] + \int_{C_2} [(xy+y^2)\,dx + x^2 dy]$$

[For C_1, $y = x^2$, $dy = 2x\,dx$ & $x=0$ to $x=1$

Also C_2, $y = x$, $dy = dx$ & $x=1$ to $x=0$.

$$= \int_{x=0}^{1} [(x\cdot x^2 + x^4)\,dx + x^2\cdot 2x\,dx] + \int_{x=1}^{0} [(x\cdot x + x^2)\,dx + x^2\,dx]$$

$$= \int_{x=0}^{1} [3x^3 + x^4]\,dx - \int_{x=0}^{1} 3x^2\,dx.$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 - \left[x^3\right]_0^1 = \frac{3}{4} + \frac{1}{5} - 1 = \frac{19-20}{20} = \frac{-1}{20}. \rightarrow (3)$$

Hence from (2) & (3). Green's Theorem Verified.

Ex-2: Verify Green's Theorem in the plane for $\int_C [(3x^2 - 8y^2)\,dx + (4x - 6xy)\,dy]$ where C is the region bdd by the parabolas $y = \sqrt{x}$ & $y = x^2$. (AKTU-2011).

Ex-3: Verify Green's Theorem in plane for $\oint_C [(x^2 - 2xy)\,dx + (x^2 y + 3)\,dy]$ where C is the boundary of the region defined by $y^2 = 8x$ & $x = 2$. (AKTU-2014).

Ex-4: Verify Green's theorem by evaluating $\int_C [(x^3 - xy^3)\,dx + (y^2 - 2xy)\,dy]$ where C is the square having the vertices at the points $(0,0)$, $(2,0)$, $(2,2)$ & $(0,2)$.

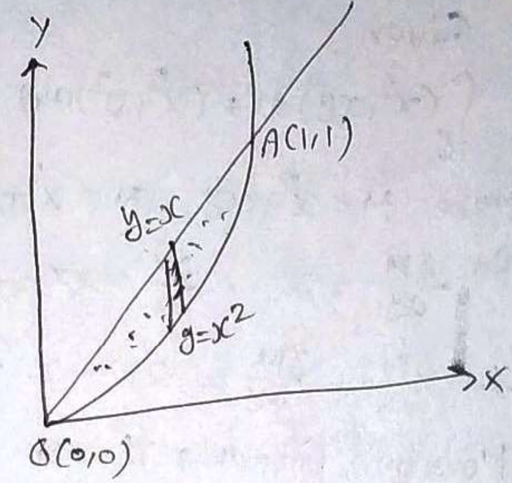
Ex-5: Verify Green's theorem for $\int_C [(x^2 + 2xy)\,dx + (y^2 + x^3 y)\,dy]$ where C is the boundary of a square with vertices $A(0,0)$; $B(1,0)$ $D(1,1)$ & $E(0,1)$. [AKTU-2009]

Hint: $\int_0^1 \int_0^1$.

(Type-II)

Ex-1→ Using Green's Theorem to evaluate $\int_C (2y^2 dx + 3x\,dy)$ where C is the boundary of the closed region bounded by $y=x$ & $y=x^2$. (AKTU - 2011, 2016, 2018).

Sol^n→ By Green's Theorem we have



$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Given $\int_C 2y^2 dx + 3x\,dy$.

Let $M = 2y^2$, $N = 3x$.

$\therefore \frac{\partial M}{\partial y} = 4y$, $\frac{\partial N}{\partial x} = 3$.

& $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3 - 4y$.

Hence by Green's theorem,

$$\oint_C M dx + N dy = \int_C 2y^2 dx + 3x\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (3-4y)\,dy\,dx.$$

$$= \int_{x=0}^{1} \left[3y - 4\frac{y^2}{2}\right]_{x^2}^{x} dx.$$

$$= \int_{x=0}^{1} (3x - 2x^2 - 3x^2 + 2x^4)\,dx.$$

$$= \left[3\frac{x^2}{2} - \frac{2}{3}x^3 - x^3 + 2\frac{x^5}{5}\right]_0^1$$

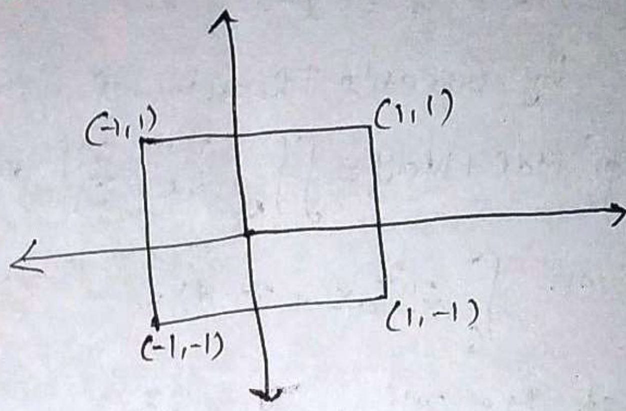
$$= \frac{3}{2} - \frac{2}{3} - 1 + \frac{2}{5} = \frac{45 - 20 - 30 + 12}{30}$$

$$= \frac{7}{30}.$$

Ex-2→ Evaluate $\int_C [3x^2 - 8y^2]\,dx + (4y - 6xy)\,dy$, where C is the boundary of the area enclosed by ~~x-axis &~~ the lines $x=0$, $y=0$, $x+y=1$.

Ex-3: Use Green's theorem to evaluate $\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $y=\pm 1$, $x=\pm 1$. [AKTU-2015, 2018].

Sol^n:



Given

$$\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$$

Here $M = x^2+xy$, $N = x^2+y^2$

then $\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 2x$.

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x.$$

Now by Green's theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\,dy.$$

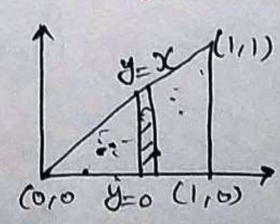
$$\Rightarrow \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{x=-1}^{1}\int_{y=-1}^{1} x\,dy\,dx.$$

$$= \int_{x=-1}^{1} x\,[y]_{-1}^{1}\,dx.$$

$$= 2\int_{x=-1}^{1} x\,dx = 2\left[\frac{x^2}{2}\right]_{-1}^{1} = \left[1^2-(-1)^2\right] = 1-1=0$$

Ex-4: Using Green's theorem, evaluate $\int_C (x^2y\,dx + x^2\,dy)$ where C is the boundary described counter clock wise of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. (AKTU-2010).

Hint: $\therefore \int_{x=0}^{1}\int_{y=0}^{x} (2x-x^2)\,dy\,dx = \frac{5}{12}$.

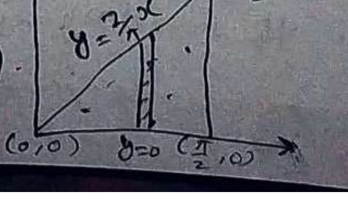


Ex-5: Evaluate $\int_C [(y-\sin x)\,dx + \cos x\,dy]$

where C is the triangle formed by $y=0$, $x=\frac{\pi}{2}$, $y=\frac{2}{\pi}x$.

Hint:

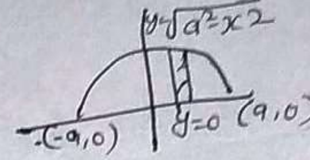
$$-\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2}{\pi}x} (\sin x+1)\,dy\,dx = -\left(\frac{\pi}{4}+\frac{2}{\pi}\right).$$



Ex-6: Apply Green's theorem to evaluate

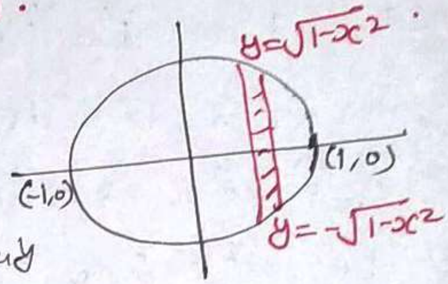
$$\int_C [(2x^2-y^2)\,dx + (x^2+y^2)\,dy]$$ where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2+y^2=a^2$.

Hint $\displaystyle\int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (2x+2y)\,dy\,dx = \frac{4}{3}a^3.$



Ex-7: Evaluate by Green's theorem $\int_C (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$ where C is the circle $x^2+y^2=1$. (AKTU-2015).

Solⁿ:



Given

$$\int_C [(\cos x \sin y - xy)\,dx + \sin x \cos y\,dy].$$

Here $M = \cos x \sin y - xy$, $N = \sin x \cos y$

$$\frac{\partial M}{\partial y} = \cos x \cos y - x, \quad \frac{\partial N}{\partial x} = \cos x \cos y.$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \cos x \cos y - \cos x \cos y + x = x.$$

Now by Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$$\Rightarrow \int_C [(\cos x \sin y - xy)\,dx + \sin x \cos y\,dy] = \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx.$$

$$= 2\int_{x=-1}^{1} x\left[\int_{y=0}^{\sqrt{1-x^2}} dy\right] dx.$$

$$= 2\int_{x=-1}^{1} x\sqrt{1-x^2}\,dx.$$

$$= 0$$

by $\longrightarrow$ $\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(-x)=f(x); \; = 0 \text{ if } f(-x) = -f(x)\right]$

$[\because$ Here $f(-x) = -f(x)]$.

Ex-8: Evaluate by Green's theorem

$$\int_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy,$$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$ & $(0,1)$.

[Ans $\pi[\cosh 1 - 1]$.

Ex-9: A vector field $\vec{F}$ is given by

$\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$. Evaluate the integral $\int_C \vec{F}\cdot d\vec{r}$ where C is the circular path given by $x^2+y^2=a^2$.

Sol^n: Given $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$

$$\& \; d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\therefore \vec{F}\cdot d\vec{r} = \sin y\,dx + x(1+\cos y)\,dy.$$

Let $M = \sin y$, $N = x + x\cos y$.

Then $\frac{\partial M}{\partial y} = \cos y$, $\frac{\partial N}{\partial x} = 1 + \cos y$.

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (1+\cos y) - \cos y = 1.$$

Now by green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_{\text{over circle}} dx\,dy = \text{Area of circle} = \pi a^2.$$

Ex-10: Using Green's theorem, evaluate the integral

$\oint_C (xy\,dy - y^2\,dx)$, where C is the square cut from the first quadrant by the lines $x=1$, $y=1$. Ans $-\frac{3}{2}$ (AKTU-2013).

MATHEMATICS-I
KAS-103T
* Lecture No- 49 *

Module-V
Vector Calculus
[redacted]

Gauss Divergence Theorem (Relation b/w surface and volume integral)

If $\vec{F}$ is a vector point funch having continuous first order partial derivatives in the region V bounded by a closed surface S, then

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV.$$

(AKTU-2012, 2017).

where $\hat{n}$ is the outward unit normal vector to the surface S.

Ex-1 Verify divergence theorem for $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$ taken over the cube bounded by the planes $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$. [AKTU-2010, 2019]

Soln By Gauss's Divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\,dV.$$

Given $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$

$$\therefore \operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y.$$

$$\therefore \iiint_V \operatorname{div}\vec{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z-y)\,dz\,dy\,dx$$

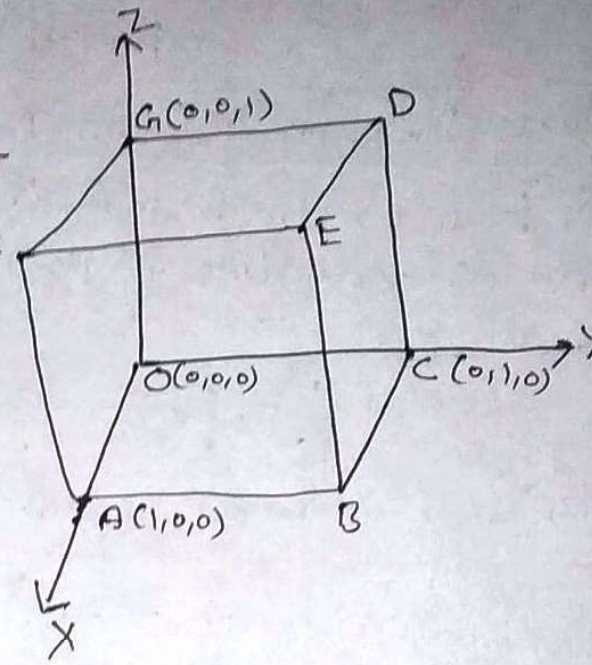
$$= \int_{x=0}^{1}\int_{y=0}^{1}\left[2z^2 - yz\right]_0^1 dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}(2-y)\,dy\,dx.$$

$$= \int_{x=0}^{1}\left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \frac{3}{2}\int_{x=0}^{1} dx = \frac{3}{2}. \Rightarrow \boxed{\iiint_V \operatorname{div}\vec{F}\,dV = \frac{3}{2}}$$

To evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, Here S is the surface of the cube bounded by the 6 faces :



⊛ for the ABEF,

$x=1$, $dx=0$ & $\hat{n}=\hat{i}$ & $dS=dy\,dz$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_{y=0}^{1}\int_{z=0}^{1} 4z\,dy\,dz = 2$$

⊛ for the face OCDG,

$x=0$, $dx=0$, $\hat{n}=-\hat{i}$, $dS=dy\,dz$

$$\iint \vec{F}\cdot\hat{n}\,dS = \int_{y=0}^{1}\int_{z=0}^{1} 0\,dy\,dz = 0.$$

⊛ for the face , BCDE, $y=1$, $dy=0$ & $\hat{n}=\hat{j}$ & $dS=dx\,dz$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_{x=0}^{1}\int_{z=0}^{1} (-1)\,dx\,dz = -1.$$

⊛ for the face, AOGF, $y=0$, $dy=0$, $\hat{n}=-\hat{j}$ & $dS=dx\,dz$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_{x=0}^{1}\int_{z=0}^{1} 0\,dx\,dz = 0$$

⊛ for the face, DEFG, $z=1$, $dz=0$, $\hat{n}=\hat{k}$, $dS=dx\,dy$.

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1} y\,dx\,dy = \frac{1}{2}$$

⊛ for the face OABC, $z=0$, $dz=0$, $\hat{n}=-\hat{k}$, $dS=dx\,dy$

$$\therefore \iint \vec{F}\cdot\hat{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1} 0\,dx\,dy = 0.$$

Hence $\iint_S \vec{F}\cdot\hat{n}\,dS = 2+0-1+0+\frac{1}{2}+0 = \frac{3}{2} \longrightarrow (2)$

Hence $\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \operatorname{div}\vec{F}\cdot dV$

Then Gauss divergence theorem verified.

Ex-2→ Verify divergence theorem $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

(AKTU-2007, 2013)

Ex-3→ Verify divergence theorem for $\vec{F} = (x^3 - yz)\hat{i} - 2x^2y\hat{j} + 2\hat{k}$ taken ove the cube bounded by the planes $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$. (AKTU- 2009, 2017).

Ex-4→ Verify Gauss's divergence theorem for

$\vec{F} = (x^3 - yz)\hat{i} + (y^3 - zx)\hat{j} + (z^3 - xy)\hat{k}$ taken over the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$ & $z=0$, $z=1$. (AKTU-2006, 2018).

Ex-5→ Verify the divergence theorem for

$\vec{F} = (2x - z)\hat{i} + x^2y\hat{j} - xz^2\hat{k}$ taken over the region bdd by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Ex-6→ Verify divergence theorem for

$\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

[AKTU-2008, 2018]

Ans $abc(a+b+c)$.

Ex-7→ The vector field $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ is defined over the volume of the cuboid given by

$0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ enclosing the surface S, evaluate $\iint_S \vec{F} \cdot \vec{ds}$.

[Hint→ $\iint_S \vec{F} \cdot \vec{ds} = \iiint_V \text{div}\,\vec{F}\, dV = abc\left(a + \frac{b}{2}\right)$.]

Ex-8→ For any closed surface S, prove that

$$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, dS = 0.$$

Soln→ By Gauss Divergence theorem, we have

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \text{div}\,\vec{F}\, dV$$

Put $\vec{F} = \text{Curl}\,\vec{F}$, we get

$$\iint_S \text{Curl}\,\vec{F} \cdot \hat{n}\, dS = \iiint_V \text{div}(\text{Curl}\,\vec{F})\, dV = 0 \quad [\because \text{div}(\text{Curl}\,\vec{F}) = 0]$$

Ex-9: Evaluate $\iint_S \vec{r} \cdot \hat{n} \, dS$, where S is closed surface and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. [AKTU- 2007]

Solⁿ: By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV$$

Put $\vec{F} = \vec{r}$.

$$\iint_S \vec{r} \cdot \hat{n} \, dS = \iiint_V \operatorname{div} \vec{r} \, dV \qquad [\because \operatorname{div} \vec{r} = 3].$$

$$= 3 \iiint_V dV$$

$$= 3V.$$

Ex-10: If S is any closed surface enclosing volume V & $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, prove that $\iint_S \vec{F} \cdot \hat{n} \, dS = 6V$.

Ex-11: If $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$, a, b, c are constants show that $\iint_S \vec{F} \cdot \vec{dS} = \frac{4}{3}\pi(a+b+c)$ where S is the surface of a unit sphere.

Ex-12: Evaluate $\int_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot \vec{dS}$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in first octant. (AKTU-2015)

(Ans 0)

Ex-13: Using the divergence theorem, evaluate the surface integral $\iint_S (yz \, dy dz + zx \, dz dx + xy \, dx dy)$ where $S: x^2 + y^2 + z^2 = 4$.

Solⁿ: 11: Given $\vec{F} = ax\hat{i} + by\hat{j} + cz\hat{k}$ then

$$\operatorname{div} \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (ax\hat{i} + by\hat{j} + cz\hat{k})$$

$$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a+b+c.$$

Now by Gauss's Theorem, $\left[\because \hat{n} = \frac{\vec{dS}}{dS}\right]$

$$\iint_S \vec{F} \cdot \vec{dS} = \iint_S \vec{F} \cdot \hat{n} \, dS$$

$$= \iiint_V \operatorname{div} \vec{F} \, dV = (a+b+c) \iiint_V dV$$

$$= (a+b+c) \times (\text{Volume of sphere})$$

$$= (a+b+c) \times \frac{4}{3}\pi \, 1^3 = \frac{4}{3}\pi(a+b+c).$$

Ex-14 → find $\iint_S \vec{F}\cdot\hat{n}\,dS$ where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and S is the surface of the sphere having centre at (3, -1, 2) and radius 3. (AKTU-2007)

Solⁿ → Given $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$.

then $\text{div}\vec{F} = \nabla\cdot\vec{F}$

$$= \frac{\partial}{\partial x}(2x+3z) + \frac{\partial}{\partial y}(-xz-y) + \frac{\partial}{\partial z}(y^2+2z)$$

$$= 2-1+2 = 3.$$

Then by Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \text{div}\vec{F}\,dV = 3\iiint_V dV = 3V$$

$$= 3 \times \text{Volume of sphere}$$

$$= 3\times\frac{4}{3}\pi(3)^3$$

$$= 108\pi.$$

Ex-15 → Use divergence theorem to evaluate the surface integral $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ where S is the portion of the plane $x+2y+3z=6$ which lies in the first octant.

[Hint – $3\iiint_V dV = 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\int_{z=0}^{\frac{6-x-2y}{3}} dz\,dy\,dx = 18$].

Ex-16 → Find $\iint_S \vec{F}\cdot\hat{n}\,dS$ by ~~verify~~ divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bdd by the cylinder $x^2+y^2=4$, $z=0$, $z=3$.

[Hint $\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{2}(4-4y+2z)\,dz\,dy\,dx = 84\pi$]

Ex-17 → Evaluate $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot\hat{n}\,ds$ where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane & bdd by this plane.

(AKTU-2014).

Sol^n Let V be the volume enclosed by the surface S. Then by divergence theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot\hat{n}\,dS$$

$$= \iiint_V \operatorname{div}(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(x^2y^2)\right]dV$$

$$= \iiint_V 2zy^2\,dV$$

$$= 2\iiint_V zy^2\,dV.$$

Changing to spherical polar coordinates

put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

& $0 \le r \le 1$; $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le 2\pi$.

$$\therefore\ 2\iiint_V zy^2\,dV = 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r\cos\theta)(r^2\sin^2\theta\sin^2\phi) \times r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^5\sin^3\theta\cos\theta\sin^2\phi\,dr\,d\theta\,d\phi$$

$$= 2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\sin^3\theta\cos\theta\sin^2\phi\left[\frac{r^6}{6}\right]_0^1 d\theta\,d\phi$$

$$= \frac{1}{3}\int_{\phi=0}^{2\pi}\left[\int_{\theta=0}^{\pi/2}\sin^3\theta\cos\theta\,d\theta\right]\sin^2\phi\,d\phi$$

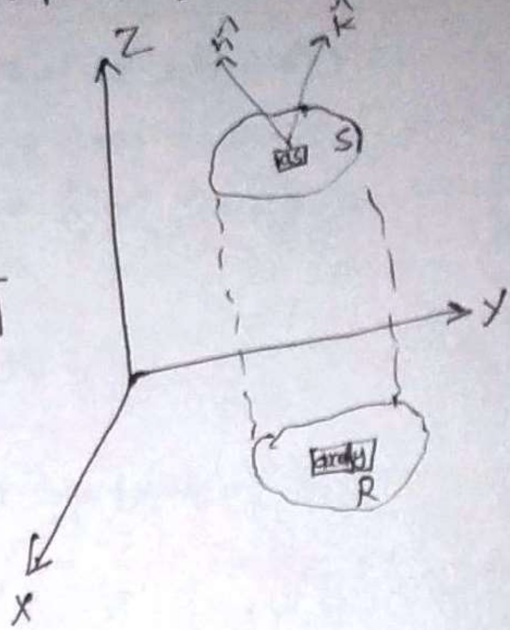
$$= \frac{1}{3}\int_{\phi=0}^{2\pi}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\sin^2\phi\,d\phi$$

$$= \frac{1}{12}\int_{\phi=0}^{2\pi}\sin^2\phi\,d\phi = \frac{1}{12}\int_{\phi=0}^{2\pi}\frac{(1-\cos 2\phi)}{2}\,d\phi$$

$$= \frac{1}{24}\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{2\pi} = \frac{\pi}{12}.\ \text{(Ans)}$$

Surface integral →

A surface integral is evaluated by expressing it as double integral over the region R.



If R be the orthogonal projection of S on xy-plane,

then $$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Similarly, taking projection on yz plane

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

taking projection on zx-plane,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$$

when $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ if

Surface is $\phi(x,y,z) = 0$.

V.imp

Note → Projection for xy-plane	Projection for yz-plane	Project for zx-plane
$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$	$dS = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$	$dS = \frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$

MATHEMATICS-I

KAS-103T

[redacted]

* Lecture - No - 50 *

Stoke's Theorem → (Relation b/w line and surface integrals) .

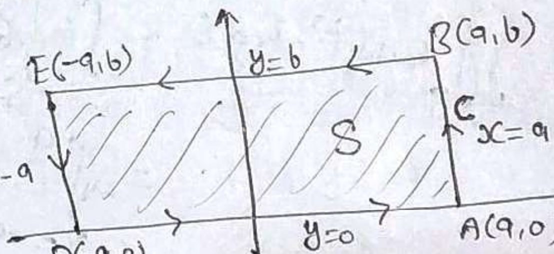
If S is an open surface bounded by a closed curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is any vector point funcⁿ having continuous first order partial derivatives, then

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,dS.}$$

where $\hat{n}$ is a unit normal vector at any point of S. (AKTU -2012 2015)

Ex-1: Verify Stoke's theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$. (AKTU-2015, 2018).

Solⁿ:



By Stoke's Theorem we have

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,dS.$$

Given $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ & $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$.

$\therefore \vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy \longrightarrow (1)$

Here curve C consists of four lines, AB, BE, ED & DA.

- for AB, $x = a$, $dx = 0$ & $y = 0$ to $y = b$
- for BE, $y = b$, $dy = 0$ & $x = a$ to $x = -a$
- for ED, $x = -a$, $dx = 0$ & $y = b$ to $y = 0$
- for DA, $y = 0$, $dy = 0$ & $x = -a$ to $x = a$.

Now $$\oint_C \vec{F}\cdot d\vec{r} = \int_{AB+BE+ED+DA} \vec{F}\cdot d\vec{r} = \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BE}\vec{F}\cdot d\vec{r} + \int_{ED}\vec{F}\cdot d\vec{r} + \int_{DA}\vec{F}\cdot d\vec{r}.$$

$$= \int_{y=0}^{b}(-2ay)dy + \int_{x=a}^{-a}(x^2+b^2)dx + \int_{y=b}^{0}(2ay\,dy) + \int_{x=-a}^{a} x^2 dx.$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^{-a} + 2a\left[\frac{y^2}{2}\right]_b^0 + \left[\frac{x^3}{3}\right]_{-a}^a$$

$$= -ab^2 + \left[-\frac{a^3}{3} - b^2a - \frac{a^3}{3} - b^2a\right] + a(0-b^2) + \frac{a^3}{3} + \frac{a^3}{3}.$$

$$= -4ab^2 \longrightarrow (2)$$

Now $\text{Curl } \vec{F} =$ or $\nabla \times \vec{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= -4y\hat{k}.$$

For the surface S in xy-plane, the unit normal is along z axis i.e $\hat{n} = \hat{k}$.

$\therefore$ $\text{Curl } \vec{F} \cdot \hat{n} = -4y\hat{k}\cdot\hat{k} = -4y$. & $dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = dx\,dy$.

Now $$\iint_S \text{Curl } \vec{F}\cdot\hat{n}\, dS = \iint_S (-4y)\, dx\,dy$$

$$= -4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dy\,dx$$

$$= -4\int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx.$$

$$= -2b^2\int_{x=-a}^{a} dx = -2b^2\left[x\right]_{-a}^{a}$$

$$= -2b^2[a-(-a)]$$

$$= -4ab^2 \longrightarrow (3)$$

from (2) & (3),

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\hat{n}\, dS}$$ Hence Stoke's Theorem Verified.

Ex-2: Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem where.

$\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices at (0,0,0), (1,0,0) and (1,1,0).

(AKTU–2008, 2012, 2014).

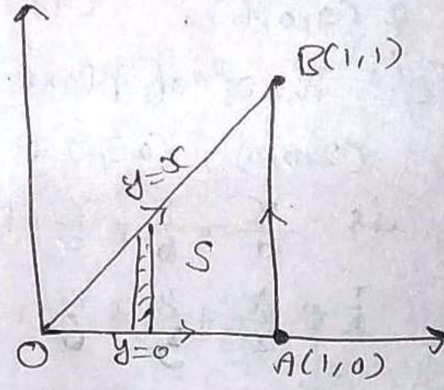
Sol^n: Given $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$

Then $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x-z) - \frac{\partial}{\partial z}(x^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x-z) - \frac{\partial}{\partial z}(y^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right]$$

$$= \hat{i}(0-0) - \hat{j}(-1-0) + \hat{k}(2x-2y)$$

$$= +\hat{j} + 2(x-y)\hat{k} \longrightarrow (1)$$



Given points are

$(0,0,0)$, $(1,0,0)$ & $(1,1,0)$.

Since z-coordinates of each vertex of the triangle is zero, then, the triangle lies in xy-plane and

$\hat{n} = \hat{k}$.

Then $\text{curl}\,\vec{F}\cdot\hat{n} = [\hat{j} + 2(x-y)\hat{k}]\cdot\hat{k}$ (As $z=0$).

$= 2(x-y)$

& $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \dfrac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = dx\,dy$ $[\hat{k}\cdot\hat{k}=1]$.

Now by Stoke's theorem,

$$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,dS$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)\,dy\,dx$$

$$= 2\int_{x=0}^{1}\left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_{x=0}^{1}\frac{x^2}{2}\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= 1/3$$

$$\Rightarrow \boxed{\oint_C \vec{F}\cdot d\vec{r} = \frac{1}{3}}$$

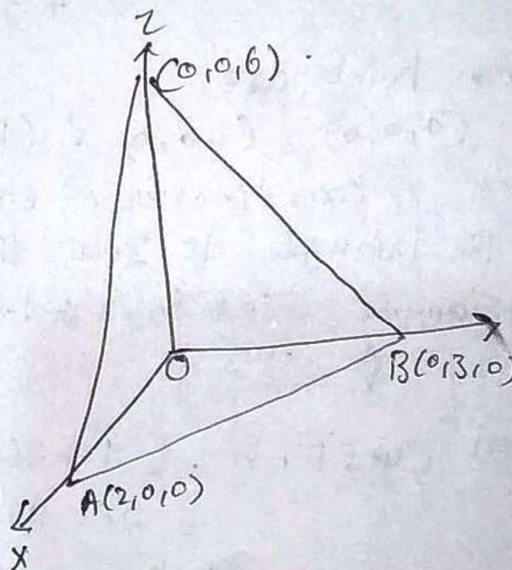
Ex-3: Verify Stoke's theorem for the funcⁿ $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square whose sides are $x = a$, $y = 0$, $x = a$, $y = a$ in the plane $z = 0$. (AKTU-2013).

Ex-4: Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ ~~$\vec{F} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$~~ over the surface of rectangle in the plane $z = a$ & bdd by $x = 0$, $y = 0$, $x = a$, $y = b$.

Ex-5: Apply Stoke's theorem to evaluate

$$\int_C [(x+y)dx + (2x - z)dy + (y+z)dz]$$

, where C is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$. (AKTU-2009).



Solⁿ: The eqⁿ of plane with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

i.e. $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \longrightarrow (1)$.

Given

$$\oint_C \vec{F}\cdot d\vec{r} = \int_C [(x+y)\,dx + (2x - z)\,dy + (y+z)\,dz]$$

then $\vec{F} = (x+y)\hat{i} + (2x - z)\hat{j} + (y+z)\hat{k}$.

Now

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x - z & y+z \end{vmatrix} = 2\hat{i} + 0\hat{j} + \hat{k}.$$

From (1), let $\phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1 = 0$.

$$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}$$

$$\therefore \text{Unit normal } \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}}{\sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}}}$$

$$\Rightarrow \hat{n} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}).$$

Let projection of this plane in xy-plane, then

$\text{Curl}\,\vec{F}\cdot\hat{n} = (2\hat{i}+\hat{k})\cdot\dfrac{(3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{14}} = \dfrac{6+1}{\sqrt{14}} = \dfrac{7}{\sqrt{14}}$

& $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \dfrac{dx\,dy}{\dfrac{(3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{14}}\cdot\hat{k}} = \dfrac{dx\,dy}{\frac{1}{\sqrt{14}}}$

Then by stoke theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,dS$$

$$= \iint_S \frac{7}{\sqrt{14}}\times\frac{dx\,dy}{\frac{1}{\sqrt{14}}}$$

$$= 7\iint_{R \text{ is } xy\text{-plane}} dx\,dy$$

[for xy-plane $z=0$, then $\frac{x}{2}+\frac{y}{3}=1$

Put $y=0$, $x=2$

& $y = 3\left(1-\frac{x}{2}\right)$.

$$= 7\int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} dy\,dx$$

$$= 7\int_{x=0}^{2}\left[y\right]_0^{\frac{6-3x}{2}} dx$$

$$= 7\int_{x=0}^{2}\frac{6-3x}{2}\,dx = \frac{7}{2}\left[6x-\frac{3x^2}{2}\right]_0^2$$

$$= \frac{7}{2}\times\left(12-\frac{12}{2}\right) = 21.$$

Ex-6: Use stoke theorem, to evaluate

$\int_C [(x+2y)\,dx + (x-z)\,dy + (y-z)\,dz]$ where C is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$ oriented in the anti-clockwise direction.

Ex-7: Verify Stoke's theorem for $\vec{F} = (2y+z,\, x-z,\, y-x)$ taken over the triangle ABC cut from the plane $x+y+z=1$ by the coordinate planes. (AKTU-2017).

Ex-8: Evaluate by Stoke's theorem $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0\le x\le\pi$, $0\le y\le 1$, $z=3$.

[Ans -2]

Ex-9: Verify Stoke's theorem for $\vec{F} = y\hat{i}+z\hat{j}+x\hat{k}$ and surface S is the portion of the sphere $x^2+y^2+z^2=1$ above the xy-plane. (AKTU-2015).

Soln 9: Given $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k} \rightarrow ①$

In xy-plane, the portion of sphere $x^2+y^2+z^2=1$ is a unit circle $x^2+y^2=1$, $z=0$.

Now
$$\vec{F}\cdot\vec{dr} = (y\hat{i} + z\hat{j} + x\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$
$$= y\,dx + z\,dy + x\,dz.$$
$$= y\,dx. \qquad [\text{For } xy\text{-plane, put } z=0,\ dz=0$$

Let $x = \cos t$, $y = \sin t$ then for circle $t=0$ to $t=2\pi$.

$\Rightarrow dx = -\sin t\,dt$, $dy = \cos t\,dt$.

$$\therefore \oint_C \vec{F}\cdot\vec{dr} = \int_{t=0}^{2\pi} y\,dx = \int_{t=0}^{2\pi} \sin t(-\sin t)\,dt$$
$$= -\int_{t=0}^{2\pi} \sin^2 t\,dt$$
$$= -\int_{t=0}^{2\pi}\left[\frac{1-\cos 2t}{2}\right]dt$$
$$= -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$
$$= -\frac{1}{2}\left[2\pi - \frac{\sin 4\pi}{2} - 0\right]$$
$$= -\pi. \longrightarrow ②$$

Again
$$\operatorname{Curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}.$$

Let $\phi = x^2+y^2+z^2-1=0$

then
$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$
$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\hat{i} + y\hat{j} + z\hat{k}.$$
$$\therefore \operatorname{Curl}F\cdot\hat{n} = (-\hat{i}-\hat{j}-\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k}) = -x-y-z$$

$$\cancel{ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k}|} = \frac{dx\,dy}{z}}$$

Then
$$\iint_S \operatorname{Curl}\vec{F}\cdot\hat{n}\,dS = \iint_S (-x-y-z)\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \longrightarrow ③.$$

Using spherical Polar coordinates,

$x = r\sin\theta\cos\phi \;,\; y = r\sin\theta\sin\phi \;,\; z = r\cos\theta$

$\Rightarrow x = \sin\theta\cos\phi \;,\; y = \sin\theta\sin\phi \;,\; z = \cos\theta$

~~$\iint dx\,dy = \iint r^2\sin\theta\, d\theta\, d\phi = \sin\theta\, d\theta\, d\phi$~~

& $[dS = \sin\theta\, d\theta\, d\phi]$ & for sphere (hemi) $\theta = 0$ to $\theta = \frac{\pi}{2}$ & $\phi = 0$ to $\phi = 2\pi$.

Using in ①, we get

$$\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\, dS = -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}[\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta)\sin\theta\, d\theta\, d\phi$$

$$= -\int_{\theta=0}^{\pi/2}\Big[\sin\theta\sin\phi - \sin\theta\cos\phi + \cos\theta \times \phi\Big]_0^{2\pi}\sin\theta\, d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\big[\sin\theta(\sin 2\pi - \sin 0) - \sin\theta(\cos 2\pi - \cos 0) + \cos\theta(2\pi - 0)\big]\sin\theta\, d\theta$$

$$= -2\pi\int_{\theta=0}^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= -\pi\int_{\theta=0}^{\pi/2}\sin 2\theta\, d\theta = -\pi\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

~~$= +\pi\left[-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2}\right]_0^{\pi}$~~

$$= -\pi\left[-\frac{\cos\pi}{2} + \frac{\cos 0}{2}\right] = -\pi.$$

Hence Stoke's theorem verified.

Ex-10 → Verify stokes theorem for the vector field $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half surface of $x^2+y^2+z^2=1$, bounded by its projection on x-y-plane.

Hint → [$\text{Curl}\,\vec{F} = \hat{k}$ then $\text{Curl}\,\vec{F}\cdot\hat{n} = \hat{n}\cdot\hat{k}$

& $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Hence $\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\, dS = \iint_R dx\,dy$.]

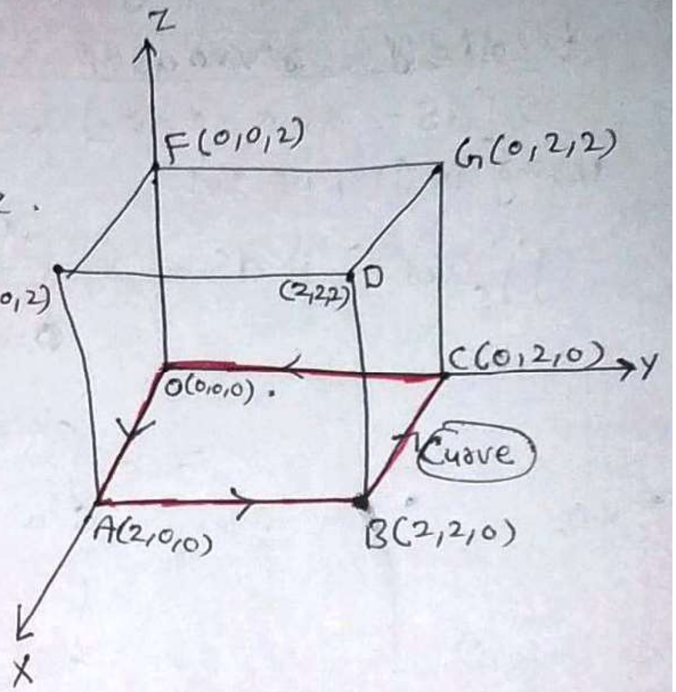
[Ans → π] .

Ex-11: Verify Stoke's Theorem for $\vec{F} = (y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$ over the surface of cube $x=0, y=0, z=0, x=2, y=2, z=2$. above the XOY plane (open the bottom). (AKTU-2014).

Sol^n:

Given $\vec{F} = (y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$

$\therefore \vec{F}\cdot\vec{dr} = (y-z+2)dx + (yz+4)dy - xz\,dz$.



for the curve at bottom,

- Along OA,
  $y=0, z=0$ & $dy=0, dz=0$
  & $x=0$ to $x=2$.
- Along AB,
  $x=2, z=0$ then $dx=0, dz=0$
  & $y=0$ to $y=2$.
- Along BC,
  $y=2, z=0 \Rightarrow dy=0, dz=0$
  & $x=2$ to $x=0$.
- Along CO, $x=0, z=0 \Rightarrow dx=0, dz=0$
  & $y=2$ to $y=0$.

Now $\oint_C \vec{F}\cdot\vec{dr} = \int_{OA+AB+BC+CO} \vec{F}\cdot\vec{dr}$

$$= \int_{OA}\vec{F}\cdot\vec{dr} + \int_{AB}\vec{F}\cdot\vec{dr} + \int_{BC}\vec{F}\cdot\vec{dr} + \int_{CO}\vec{F}\cdot\vec{dr}$$

$$= \int_{x=0}^{2} 2\,dx + \int_{y=0}^{2} 4\,dy + \int_{x=2}^{0} 4\,dx + \int_{y=2}^{0} 4\,dy$$

$$= 4 + 8 - 8 - 8 = -4.$$

$$\Rightarrow \boxed{\oint_C \vec{F}\cdot\vec{dr} = -4} \longrightarrow (1).$$

Now $\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix} = -y\hat{i} + (z-1)\hat{j} - \hat{k}$.

Here we have to integrate over the five surfaces ABDE, OCGF, BCGD, OAEF, DEFG.

(i) for the face, ABDE, $x=2$, $\hat{n}=\hat{i}$ & $dS = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = dy\,dz$

$$\therefore \iint \text{Curl}\,\vec{F}\cdot\hat{n}\,dS = \iint (-y)\,dy\,dz$$

$$= -\int_{y=0}^{2}\int_{z=0}^{2} dy\,dz = -4$$

(ii) for the face, OCGF, $(x=0)$, $\hat{n}=-\hat{i}$, $dS = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = dy\,dz$

$$\therefore \iint \text{Curl}\,\vec{F}\cdot\hat{n}\,dS = \iint y\,dy\,dz = \int_{y=0}^{2}\int_{z=0}^{2} y\,dy\,dz = 4.$$

(iii) for the face, BCGD, $y=2$, $\hat{n}=\hat{j}$ & $dS = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = dx\,dz$

$$\therefore \iint \text{Curl}\,\vec{F}\cdot\hat{n}\,dS = -\iint (z-1)\,dx\,dz$$

$$= -\int_{x=0}^{2}\int_{z=0}^{2} (z-1)\,dx\,dz = 0$$

(iv) for the face OAEF, $(y=0)$, $\hat{n}=-\hat{j}$, $dS = dx\,dz$

$$\therefore \iint \text{Curl}\,\vec{F}\cdot\hat{n}\,dS = -\iint (z-1)\,dx\,dz = -\int_{x=0}^{2}\int_{z=0}^{2} (z-1)\,dx\,dz = 0$$

(v) for the face DEFG, $z=2$, $\hat{n}=\hat{k}$, $dS = dx\,dy$

$$\therefore \iint \text{Curl}\,\vec{F}\cdot\hat{n}\,dS = -\iint dx\,dy$$

$$= -\int_{x=0}^{2}\int_{y=0}^{2} dx\,dy = -4.$$

Hence for total surface i.e five faces

$$\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,dS = -4+4+0+0-4 = -4 \rightarrow (2)$$

From (1), & (2),

Stoke's Theorem Verified.