Engineering Mathematics - II

KAS-203T

Module - II

Multivariable Calculus - II

* Lecture - No - 15 *

[redacted]

BETA FUNCTION →

If $m, n > 0$, the the definite integral

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$ (AKTU-2018)

is called Beta Function.

Properties of Beta function →

① Symmetry of Beta function →

Prove that $\beta(m,n) = \beta(n,m)$. (AKTU-2006)

Proof: We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \longrightarrow ①$$

[Put $x = 1-y$ then $dx = -dy$ when $x=0$, $y=1$; $x=1$, $y=0$]

$$= \int_1^0 (1-y)^{m-1}[1-(1-y)]^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx \qquad \left[\int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$$

$$= \beta(n,m) \qquad (\text{From } ①)$$

$$\Rightarrow \boxed{\beta(m,n) = \beta(n,m)}$$

② Transformation of Beta function →

Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$.

(It is another form of Beta func^n)

Proof: We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \longrightarrow ①$

Put $x = \frac{1}{1+y}$ then $dx = \frac{(-1)\,dy}{(1+y)^2}$ | when $x=0$ then $y=\infty$; when $x=1$ then $y=0$.

in eq^n ①, we get

$$\beta(m,n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left[1 - \frac{1}{1+y}\right]^{n-1} \frac{(-1)}{(1+y)^2}\,dy$$

$$= \int_0^{\infty} \frac{1}{(1+y)^{m-1}} \left[\frac{1+y-1}{1+y}\right]^{n-1} \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^{\infty} \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\Rightarrow \boxed{\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}} \qquad \left[\because \int_a^b f(y)\,dy = \int_a^b f(x)\,dx\right]$$

Also, $\boxed{\beta(m,n) = \beta(n,m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.}$

__Ex-1)__ Evaluate $\int_0^1 x^4 (1-\sqrt{x})^5\,dx$.

__Sol^n)__

$$\int_0^1 x^4 (1-\sqrt{x})^5\,dx$$

[Let $\sqrt{x} = t$
$\Rightarrow x = t^2$
$\Rightarrow dx = 2t\,dt$.
& when $x=0,\ t=0$
$x=1,\ t=1$]

$$= \int_0^1 (t^2)^4 (1-t)^5\, 2t\,dt$$

$$= 2\int_0^1 t^9 (1-t)^5\,dt$$

$$= 2\int_0^1 t^{10-1} (1-t)^{6-1}\,dt$$

$$= 2\,\beta(10,6).$$

__Ex-2)__ Evaluate $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$ [AKTU-2012] 2007

__Sol^n)__ By the transformation of Beta func^n, we have

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \longrightarrow (1)$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right.$$

Now

$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \left[\text{Put } x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt\right.$$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}\left(-\frac{1}{t^2}\right)dt}{\left[1+\frac{1}{t}\right]^{m+n}} \qquad \left.\begin{matrix}\text{when } x=1, & t=1 \\ x=\infty, & t=0\end{matrix}\right]$$

$$= \int_0^1 \frac{\frac{1}{t^{m+1}}\,dt}{\frac{(t+1)^{m+n}}{t^{m+n}}} = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \qquad \left[\int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right]$$

Using this value in eq^n (1), we get

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\Rightarrow \boxed{\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)}$$

(3) Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

(Anothe transformation of Beta func^n).

Proof: We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \longrightarrow (1)$

Put $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta\,d\theta$

when $x=0$, $\theta=0$

$x=1$, $\theta=\frac{\pi}{2}$: we get

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} [1-\sin^2\theta]^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \; \cos^{2n-1}\theta \cdot \sin\theta\cos\theta\, d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \; \cos^{2n-1}\theta\, d\theta}$$

Ex-1→ Prove that

$$\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\, dx$$ (AKTU-2015).

Solⁿ→

$$\text{R.H.S} = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\, dx$$

$$= a^m b^n \int_0^\infty \frac{\left(\frac{b}{a}y\right)^{m-1} \frac{b}{a}\, dy}{[by+b]^{m+n}}$$

[Put $ax = by$
$\Rightarrow x = \frac{b}{a}y$.
$\Rightarrow dx = \frac{b}{a}\, dy$
when $x=0,\ y=0$
$x=\infty,\ y=\infty$]

$$= a^m b^n \frac{\frac{b^{m-1}}{a^{m-1}} \cdot \frac{b}{a}}{b^{m+n}} \int_0^\infty \frac{y^{m-1}\, dy}{(1+y)^{m+n}}$$

$$= a^m b^n \frac{\frac{b^m}{a^m}}{b^{m+n}} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

$$= \frac{b^{m+n}}{b^{m+n}} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}$$

$$= \beta(m,n) = \text{L.H.S.}$$

Hence $$\boxed{\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\, dx}$$

Home Assignment

Ex-1→ Prove $\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\, \beta(m+1, n+1)$
[Put $x = a+(b-a)z$]

Ex-2→ Prove $\int_0^a x^{n-1} (a-x)^{m-1}\, dx = a^{m+n-1}\, \beta(m,n)$
[Put $x = ay$].

ENGINEERING MATHEMATICS-II

KAS-203T

Module-II

* Multi-Cal.-II *

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* Lecture-16 *

Gamma function →

The definite integral

$$\boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx} \text{ (func}^n \text{ of } n),$$

is called Gamma func^n and read as gamma func^n of n.

(AKTU-2006, 2008, 2010)

Properties of Gamma func^n →

① Prove that $\Gamma 1 = 1$.

Sol^n: We have $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$.

Put $n=1$, $\Gamma 1 = \int_0^\infty e^{-x} x^{1-1}\,dx = \int_0^\infty e^{-x}\,dx$

$$= -\left[e^{-x}\right]_0^\infty$$

$$= -\left[e^{-\infty} - e^0\right]$$

$$= -[0-1] = 1.$$

$$\Rightarrow \boxed{\Gamma 1 = 1}$$

② Prove that $\boxed{\Gamma n+1 = n\,\Gamma n}$

Sol^n: We have $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \longrightarrow ①$

Put $n = n+1$, we get

$$\Gamma n+1 = \int_0^\infty e^{-x} x^n\,dx$$

$$= \left[\left(x^n \frac{e^{-x}}{(-1)}\right)_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{(-1)}\,dx\right]$$

$$= 0 + n\int_0^\infty e^{-x} x^{n-1}\,dx.$$

$$= n\,\Gamma n \qquad \text{(from ①)}$$

$$\Rightarrow \boxed{\Gamma n+1 = n\,\Gamma n}$$

③ Prove that $\boxed{\Gamma(n+1) = \lfloor n}$

Sol^n: We have

$$\Gamma(n+1) = n\Gamma n \longrightarrow ①$$

Replace n by (n-1), we get

$$\Gamma n = (n-1)\,\Gamma(n-1) \longrightarrow ②$$

Put n by (n-1) in ②, we get

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2) \longrightarrow ③$$

Put n by (n-1) in ③, we get

$$\Gamma(n-2) = (n-3)\,\Gamma(n-3) \longrightarrow ④$$

and so on.

Using these values in eq^n ①, we get

$$\Gamma(n+1) = n(n-1)\,\Gamma(n-1) \quad (\text{from } ②)$$

$$= n(n-1)(n-2)\,\Gamma(n-2) \quad (\text{from } ③)$$

$$= n(n-1)(n-2)(n-3)\,\Gamma(n-3) \quad (\text{from } ④)$$

$$\vdots$$

$$= n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1\,\Gamma 1$$

$$\Rightarrow \boxed{\Gamma(n+1) = \lfloor n}$$

Transformation of Gamma func^n →

① Prove that $\Gamma n = a^n \int_0^\infty e^{-ax} x^{n-1}\,dx$ [AKTU-2016]

Sol^n: we have $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-ay} (ay)^{n-1}\, a\,dy.$$

[Put $x = ay$
$\Rightarrow dx = a\,dy$
when $x=0$, $y=0$
$x=\infty$, $y=\infty$]

$$\Rightarrow \Gamma n = a^n \int_0^\infty e^{-ay} y^{n-1}\,dy$$

$$\Rightarrow \boxed{\Gamma n = a^n \int_0^\infty e^{-ax} x^{n-1}\,dx}$$

$\left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$

or $\left(\int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^n}\right)$

② Prove that $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy = \Gamma n$.

Proof:

$$\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

Put $\log \frac{1}{y} = x$

$\Rightarrow \frac{1}{y} = e^{x}$

$\Rightarrow y = e^{-x} \Rightarrow dy = -e^{-x} dx$

when $y = 0,\ x = \infty$

$y = 1,\ x = 0$

$$= \int_{\infty}^{0} x^{n-1} (-e^{-x} dx)$$

$$= \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= \Gamma n$$

$$\Rightarrow \boxed{\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy = \Gamma n}$$

Note $\Gamma \frac{1}{2} = \sqrt{\pi}$

Ex-1: Prove that $\int_0^1 (x \log x)^4 dx = \frac{\lfloor 4}{5^5}$. [AKTU-2009]

Sol^n:

$$\int_0^1 (x \log x)^4 dx$$

Let $\log x = -t$.

$\Rightarrow x = e^{-t}$.

& $dx = -e^{-t} dt$

when $x = 0,\ t = \infty$

$x = 1,\ t = 0$

$$= \int_0^1 x^4 (\log x)^4 dx$$

$$= \int_{\infty}^{0} e^{-4t} (-t)^4 \cdot [-e^{-t} dt]$$

$$= \int_0^{\infty} e^{-5t} t^4 dt$$

$$= \int_0^{\infty} e^{-5t} t^{5-1} dt$$

$$\left[\because \int_0^{\infty} e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}\right.$$

$$\left.\& \ \Gamma n = \lfloor n-1 \right]$$

$$= \frac{\Gamma 5}{5^5} = \frac{\lfloor 4}{5^5}.$$

Ex-2: Evaluate $\int_0^{\infty} e^{-x^2} dx$. [AKTU-2011, 2018]

Sol^n:

$$\int_0^{\infty} e^{-x^2} dx$$

Put $x^2 = t \Rightarrow x = t^{1/2}$ & $dx = \frac{1}{2} t^{-1/2} dt$

when $x = 0,\ t = 0$

$x = \infty,\ t = \infty$.

$$= \int_0^{\infty} e^{-t} \frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt = \frac{1}{2} \Gamma \frac{1}{2} = \frac{\sqrt{\pi}}{2} \quad \left[\because \Gamma \tfrac{1}{2} = \sqrt{\pi}\right]$$

[Note- $\Gamma\frac{1}{2} = \sqrt{\pi}$]

Ex-3→ Evaluate $\Gamma\left(-\frac{5}{2}\right)$ [AKTU-2012, 2014]

Solⁿ→ We have $\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$ $\quad [\because \Gamma(n+1) = n\Gamma n]$

$$\Rightarrow \Gamma\frac{1}{2} = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2}\Gamma\left(-\frac{3}{2}+1\right)$$

$$= -\frac{1}{2}\cdot -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{3}{2}\Gamma\left(-\frac{5}{2}+1\right)$$

$$\Rightarrow \Gamma\frac{1}{2} = \frac{1}{2}\cdot\frac{3}{2}\cdot -\frac{5}{2}\Gamma\left(-\frac{5}{2}\right)$$

$$\Rightarrow \sqrt{\pi} = \frac{-15}{8}\Gamma\left(-\frac{5}{2}\right) \Rightarrow \boxed{\Gamma\left(-\frac{5}{2}\right) = \frac{8\sqrt{\pi}}{-15}}$$

Home Assignment

Ex-1→ Evaluate (i) $\Gamma\left(-\frac{3}{2}\right)$ Ans $\frac{4\sqrt{\pi}}{3}$ (AKTU-2008, 2015). (ii) $\Gamma\left(-\frac{1}{2}\right)$ [⇒ Ans - $-2\sqrt{\pi}$]. (AKTU-2011).

Ex-2→ Prove that $\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n}{(m+1)^{n+1}}\Gamma(n+1)$

(Hint - Put $\log x = -t$).

Ex-3→ Prove that $\int_0^\infty \sqrt{x}\, e^{-x}\,dx = \frac{1}{2}\sqrt{\pi}$ (AKTU-2014)

Ex-4→ Evaluate $\int_0^\infty x^{n-1} e^{-h^2x^2}\,dx$ [Ans - $\frac{1}{2h^n}\Gamma\frac{n}{2}$]

Ex-5→ Evaluate $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\,dx$. (AKTU-2008).

Solⁿ→ $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\,dx = \int_0^\infty (t^2)^{1/4} e^{-t}\, 2t\,dt$

Put $\sqrt{x} = t$
$\Rightarrow x = t^2$
& $dx = 2t\,dt$
when $x=0$, $t=0$
$x=\infty$, $t=\infty$ we get

$$= 2\int_0^\infty e^{-t} t^{3/2}\,dt$$

$$= 2\int_0^\infty e^{-t} t^{5/2-1}\,dt$$

$$= 2\Gamma\frac{5}{2}$$

$$= 2\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$= \frac{3}{2}\sqrt{\pi}.$$

Ex-6 → Find $\int_0^\infty x^4 e^{-x^2} dx$ (AKTU-2013) [Ans - $\frac{3\sqrt{\pi}}{8}$]

Ex-7 → Evaluate $\int_0^\infty e^{-h^2x^2} dx$ (AKTU-2011) Ans $\frac{\sqrt{\pi}}{2h}$.

[Hint Put $h^2x^2 = t$]

Ex-8 → Find $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ Ans $\sqrt{\pi}$.

[Put, $-\log x = t$]

Ex-9 → $\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}}$ [Ans $\sqrt{2\pi}$].

Ex-10 → $\int_0^1 (x \log x)^3 dx$ [Ans → $-\frac{3}{128}$].

ENGINEERING MATHEMATICS - II

KAS-203T

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* Lecture No - 17 *

Relation between Beta & Gamma function →

Prove that $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$ [AKTU - 2009, 2010, 2013, 2018]

Proof → we have $\Gamma n = a^n \int_0^\infty e^{-ax} x^{n-1} dx \longrightarrow (1)$ and $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \rightarrow (2)$

Replace a by Z in eqn (1), we get

$$\Gamma n = Z^n \int_0^\infty e^{-Zx} x^{n-1} dx$$

Multiplying both side by $e^{-Z} Z^{m-1}$, we get

$$e^{-Z} Z^{m-1} \Gamma n = e^{-Z} Z^{m-1} Z^n \int_0^\infty e^{-Zx} x^{n-1} dx.$$

$$\Rightarrow e^{-Z} Z^{m-1} \Gamma n = \int_0^\infty e^{-(1+x)Z} Z^{m+n-1} x^{n-1} dx.$$

Integrating both side w.r.t Z from 0 to ∞, we get

$$\Gamma n \int_0^\infty e^{-Z} Z^{m-1} dZ = \int_0^\infty \left[\int_0^\infty e^{-(1+x)Z} Z^{m+n-1} x^{n-1} dx \right] dZ$$

$$\Rightarrow \Gamma n \, \Gamma m = \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-(1+x)Z} Z^{(m+n)-1} dz \right] dx$$

$$= \int_0^\infty x^{n-1} \left[\frac{\Gamma m+n}{(1+x)^{m+n}} \right] dx$$

(from (1) & (2))

$$= \Gamma m+n \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\Rightarrow \Gamma n \, \Gamma m = \Gamma m+n \; \beta(m,n) \qquad \left[\because \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m}} d \right.$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{\Gamma m+n}{\Gamma m \, \Gamma n}}$$

Hence proved.

② Show that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} \quad \text{(AKTU-2004)}$$

Proof: We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

[Put $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta\,d\theta$; when $x=0$, $\theta=0$ & $x=1$, $\theta=\frac{\pi}{2}$]

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}\left[1-\sin^2\theta\right]^{n-1} \times 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot \sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \frac{1}{2}\beta(m,n) = \frac{1}{2}\frac{\Gamma m\,\Gamma n}{\Gamma m+n} \longrightarrow ①$$

Replace $2m-1=p$ & $2n-1=q$

or $m=\frac{p+1}{2}$, $n=\frac{q+1}{2}$ in eqn ①, we get

$$\boxed{\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta = \frac{1}{2}\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}}$$

③ Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$ [AKTU - 206, 2010]

Proof: We have $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} \longrightarrow ①$

Put $p=0$, $q=0$ in ①, we get

$$\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{2}{2}} = \int_0^{\pi/2} \sin^0\theta\cos^0\theta\,d\theta = \int_0^{\pi/2} d\theta$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{2\,\Gamma 1} = [\theta]_0^{\pi/2} \Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = 2\cdot\frac{\pi}{2}$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi \Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

④ Prove that $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$ $(0<n<1)$.

Solⁿ We have
$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Put $m+n=1 \Rightarrow m=1-n$ in above eqⁿ, we get
$$\beta(1-n,n) = \frac{\Gamma 1-n \, \Gamma n}{\Gamma 1}$$
$$\Rightarrow \beta(n,1-n) = \Gamma n \, \Gamma 1-n \longrightarrow ① \quad [\because \beta(m,n)=\beta(n,m)]$$

Also, we have
$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Put $m=1-n$,
$$\beta(1-n,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{1-n+n}} dx$$
$$\Rightarrow \beta(n,1-n) = \int_0^\infty \frac{x^{n-1}}{1+x} dx \longrightarrow ②$$

from ① & ②,
$$\Gamma n \, \Gamma 1-n = \int_0^\infty \frac{x^n}{1+x} dx$$
$$\Rightarrow \boxed{\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}} \quad \left[\because \int_0^\infty \frac{x^n}{1+x} dx = \frac{\pi}{\sin n\pi}\right].$$

Ex-1. Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}} dx$. [AKTU-2008]

Solⁿ
$$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}} dx = \int_0^\infty \frac{x^4}{(1+x)^{15}} dx + \int_0^\infty \frac{x^9}{(1+x)^{15}} dx.$$
$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}} dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}} dx$$
$$= \beta(5,10) + \beta(10,5) \quad \left[\because \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx\right]$$
$$= 2\beta(5,10) \quad [\because \beta(m,n) = \beta(n,m)]$$
$$= 2\frac{\Gamma 5 \, \Gamma 10}{\Gamma 5+10} = 2\frac{\lfloor 4 \, \lfloor 9}{\lfloor 14} = \frac{1}{5005}. \quad \left[\because \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}\right].$$

Ex-2: Evaluate $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$ [AKTU-2007, 2013].
[Ans - 0]

Ex-3: Using Bet and Gamma functions, evaluate

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx.$$ [AKTU-2006, 2014, 2018]

Sol^n:
Given $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx$

$$= \int_0^1 x^{3/2}(1-x^3)^{-1/2}\,dx$$

[Put $x^3 = t$
$\Rightarrow x = t^{1/3}$
$\Rightarrow dx = \frac{1}{3} t^{-2/3}\,dt.$
when $x=0,\ t=0$
$x=1,\ t=1$]

$$= \int_0^1 (t^{1/3})^{3/2}(1-t)^{-1/2}\frac{1}{3}t^{-2/3}\,dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{2}}t^{-2/3}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{3}\int_0^1 t^{-1/6}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{3}\int_0^1 t^{5/6-1}(1-t)^{\frac{1}{2}-1}\,dt.$$

$$= \frac{1}{3}\beta\left(\frac{5}{6},\frac{1}{2}\right)$$ $\left[\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx\right]$

$$= \frac{1}{3}\frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)}$$ $\left[\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$

$$= \frac{1}{3}\frac{\Gamma\frac{5}{6}\sqrt{\pi}}{\Gamma\frac{4}{3}} = \frac{1}{3}\frac{\Gamma\frac{5}{6}\sqrt{\pi}}{\frac{1}{3}\Gamma\frac{1}{3}}$$ $\left[\because \Gamma n+1 = n\Gamma n\right]$

$$= \sqrt{\pi}\frac{\Gamma\frac{5}{6}}{\Gamma\frac{1}{3}}.$$

Note: $\sqrt{\pi}\frac{\Gamma\frac{5}{6}}{\Gamma\frac{1}{3}} = \sqrt{\pi}\frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{6}\,\Gamma\frac{2}{3}}{\Gamma\frac{1}{3}\,\Gamma\frac{1}{6}\,\Gamma\frac{2}{3}} = \sqrt{\pi}\frac{\Gamma\left(1-\frac{1}{6}\right)\Gamma\frac{1}{6}\,\Gamma\frac{2}{3}}{\Gamma\left(1-\frac{1}{3}\right)\Gamma\frac{1}{3}\,\Gamma\frac{1}{6}}$

$$= \sqrt{\pi}\frac{\frac{\pi}{\sin\frac{\pi}{6}}}{\frac{\pi}{\sin\frac{\pi}{3}}}\frac{\Gamma 2/3}{\Gamma 1/6} = \sqrt{3\pi}\frac{\Gamma 2/3}{\Gamma 1/6}.$$

Ex-4: Prove that

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q} \quad (p>0, q>0)$$

[AKTU-2012, 2015]

Proof: We have $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \rightarrow (1)$

Now $\frac{\beta(p,q+1)}{q} = \frac{1}{q}\beta(p,q+1) = \frac{1}{q}\frac{\Gamma p \, \Gamma q+1}{\Gamma p+q+1}$

$$= \frac{1}{q}\frac{\Gamma p \; q\,\Gamma q}{(p+q)\,\Gamma p+q} \qquad [\because \Gamma n+1 = n\,\Gamma n]$$

$$= \frac{1}{p+q}\frac{\Gamma p \, \Gamma q}{\Gamma p+q}$$

$$= \frac{\beta(p,q)}{p+q}. \quad (\text{from } (1))$$

Also, $\frac{\beta(p+1,q)}{p} = \frac{1}{p}\beta(p+1,q)$

$$= \frac{1}{p}\frac{\Gamma p+1 \; \Gamma q}{\Gamma p+q+1} \qquad [\because \Gamma n+1 = n\,\Gamma n$$

$$= \frac{1}{p}\frac{p\,\Gamma p \; \Gamma q}{(p+q)\,\Gamma p+q}$$

$$= \frac{1}{p+q}\frac{\Gamma p \, \Gamma q}{\Gamma p+q} = \frac{\beta(p,q)}{p+q} \quad (\text{from } (1))$$

Hence $\boxed{\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}}$

Home Assignment

Ex-5: Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$ (AKTU-2007)

Ex-6: Prove that $\beta(l,m)\,\beta(l+m,n)\cdot\beta(l+m+n,p) = \frac{\Gamma l \, \Gamma m \, \Gamma n \, \Gamma p}{\Gamma l+m+n+p}$.

Ex-7: Prove $\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{396}$.

Ex-8: Evaluate $\int_0^2 x(8-x^3)^{1/3}\,dx$. [AKTU-2010]

Sol^n 8: $\int_0^2 x(8-x^3)^{1/3}\,dx$

$\left[\text{Put } x^3 = 8y \Rightarrow x = 2y^{1/3} \Rightarrow dx = 2\cdot\frac{1}{3}y^{-2/3}\,dy\right.$

when $x=0,\ y=0$

$\left.x=2,\ y=1\right]$

$$= \int_0^1 2y^{1/3}(8-8y)^{1/3}\,\frac{2}{3}y^{-2/3}\,dy$$

$$= \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}\,dy$$

$$= \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{\frac{4}{3}-1}\,dy$$

$$= \frac{8}{3}\,\beta\left(\frac{2}{3},\frac{4}{3}\right) \qquad \left[\because \beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,dy\right.$$

$$= \frac{8}{3}\,\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} \qquad \left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right.$$

$$= \frac{8}{3}\,\frac{\Gamma\frac{2}{3}\,\Gamma\left(\frac{1}{3}+1\right)}{\Gamma 2}$$

$$= \frac{8}{3}\,\frac{\Gamma\frac{2}{3}\,\frac{1}{3}\,\Gamma\frac{1}{3}}{\lfloor 1} \qquad \left[\because \Gamma(n+1) = n\,\Gamma n \ \ \&\ \ \Gamma n = \lfloor n-1\right]$$

$$= \frac{8}{9}\,\Gamma\frac{2}{3}\,\Gamma\frac{1}{3} = \frac{8}{9}\,\Gamma\frac{2}{3}\,\Gamma\frac{1}{3} = \frac{8}{9}\,\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{8}{9}\,\frac{\pi}{\sin\frac{\pi}{3}} \qquad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right.$$

$$= \frac{8}{9}\,\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{16\pi}{9\sqrt{3}}.$$

Ex-9: Evaluate $\int_0^\infty \frac{dx}{1+x^4}$ [AKTU-2012]

Sol^n: $\int_0^\infty \frac{dx}{1+x^4}$

Put $x^2 = \tan\theta$

$x = \tan^{1/2}\theta$

then $dx = \frac{1}{2}\tan^{-1/2}\theta\cdot\sec^2\theta\,d\theta$

when $x=0,\ \theta=0$

$x=\infty,\ \theta=\frac{\pi}{2}.$

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \int_0^{\pi/2} \frac{\frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta}{1+\tan^2\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2} \tan^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} \quad \left[\because \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}\right.$$

$$= \frac{1}{4} \frac{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}{\Gamma 1}$$

$$= \frac{1}{4}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{4} \frac{\pi}{\sin\frac{\pi}{4}} \quad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right.$$

$$= \frac{1}{4} \frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\pi\sqrt{2}}{4}.$$

Ex-10 : Evaluate $\int_0^1 \frac{dx}{\sqrt{1+x^4}}$ [AKTU - 2011]

Sol^n: Let $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^2 = \tan\theta \Rightarrow x = \tan^{1/2}\theta$

$\Rightarrow dx = \frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta$.

when $x = 0,\ \theta = 0$

$x = 1,\ \theta = \frac{\pi}{4}$.

$$= \int_0^{\pi/4} \frac{\frac{1}{2}\tan^{-1/2}\theta \sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta\cos\theta}} d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\frac{1}{2}(2\sin\theta\cos\theta)}} d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Put $2\theta = t$

$\theta = t/2 \Rightarrow d\theta = \frac{dt}{2}$

when $\theta = 0,\ t = 0$

$\theta = \frac{\pi}{4},\ t = \frac{\pi}{2}$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{dt}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2} t \cdot \cos^0 t \, dt$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \qquad \left[\because \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right.$$

$$= \frac{1}{4\sqrt{2}} \sqrt{\pi} \frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

$$= \frac{1}{4\sqrt{2}} \sqrt{\pi} \frac{\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\left(\Gamma\frac{1}{4}\right)^2}{\left(\pi / \sin\frac{\pi}{4}\right)}$$

$$\left[\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{8\sqrt{\pi}} \left(\Gamma\frac{1}{4}\right)^2.$$

Ex-11 Prove that the following results:

i) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$ (AKTU-2011).

Soln $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\tan\left(\frac{\pi}{2}-\theta\right)}\, d\theta \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right.$$

$$= \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta \, d\theta$$

$$= \frac{\Gamma\frac{\frac{1}{2}+1}{2}\,\Gamma\frac{-\frac{1}{2}+1}{2}}{2\,\Gamma\frac{\frac{1}{2}-\frac{1}{2}+2}{2}} = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\,\Gamma 1} \qquad \left[\because \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}\right.$$

$$= \frac{1}{2} \Gamma\frac{1}{4}\,\Gamma 1-\frac{1}{4} = \frac{1}{2} \frac{\pi}{\sin\frac{\pi}{4}} \qquad \left[\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sqrt{2}}.$$

(ii) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$. [AKTU-2014]

<u>Solⁿ</u>→

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}$$

$$= \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{2\Gamma\frac{3}{4}} \times \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{2\Gamma\frac{5}{4}}$$

$$= \frac{1}{4}\frac{\Gamma\frac{1}{4}\left(\Gamma\frac{1}{2}\right)^2}{\Gamma\frac{5}{4}} = \frac{1}{4}\pi \frac{\Gamma\frac{1}{4}}{\Gamma\left(\frac{1}{4}+1\right)} = \frac{\pi}{4}\frac{\Gamma\frac{1}{4}}{\frac{1}{4}\Gamma\frac{1}{4}} = \pi.$$

[$\because \Gamma(n+1) = n\Gamma n$].

[& $\Gamma\frac{1}{2} = \sqrt{\pi}$]

Home Assignment

<u>Ex-1</u>→ Prove that $\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \pi\sqrt{2}$.

<u>Ex-2</u>→ Prove $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^{\infty} \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$ [AKTU-2008]

<u>Ex-3</u>→ Prove $\int_0^{\infty} \frac{x\,dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$

<u>Ex-4</u>→ Prove $\int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \pi$

<u>Ex-5</u>→ Prove $\int_0^{\infty} x^2 e^{-x^4}\,dx \times \int_0^{\infty} e^{-x^4}\,dx = \frac{\pi}{8\sqrt{2}}$.

<u>Ex-6</u>→ Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{\sqrt{3}}{2}\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{\left(\Gamma\frac{1}{3}\right)^3}{2^{7/3}\pi}$.

Ex-7: Prove that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}} \left(\Gamma\frac{1}{4}\right)^2$.

Sol^n: $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Put $x^2 = \sin\theta$

$\Rightarrow x = \sin^{1/2}\theta$

$\Rightarrow dx = \frac{1}{2}\sin^{-1/2}\theta \cos\theta \, d\theta$

when $x=0$, $\theta=0$

$x=1$, $\theta=\frac{\pi}{2}$

$$= \int_0^{\pi/2} \frac{\frac{1}{2}\sin^{-1/2}\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta$$

$$= \frac{1}{2} \frac{\Gamma\frac{-\frac{1}{2}+1}{2} \Gamma\frac{0+1}{2}}{2\Gamma\frac{-\frac{1}{2}+0+2}{2}} = \frac{1}{4} \frac{\Gamma\frac{1}{4}\sqrt{\pi}}{\Gamma\frac{3}{4}}$$

$$= \frac{1}{4}\sqrt{\pi} \frac{\Gamma\frac{1}{4}\Gamma\frac{1}{4}}{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}$$

$$= \frac{1}{4}\sqrt{\pi} \frac{\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{1}{4}\sqrt{\pi} \frac{\left(\Gamma\frac{1}{4}\right)^2}{\frac{\pi}{\sin\frac{\pi}{4}}} \qquad \left[\because \Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}\right.$$

$$= \frac{1}{4\sqrt{2\pi}} \left(\Gamma\frac{1}{4}\right)^2.$$

Duplication Formula →

Prove that $\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$ where m is +ve. [AKTU - 2013, 2016]

Hence show that $\beta(m,m) = 2^{1-2m} \beta(m, \frac{1}{2})$

Proof:- We know that

$$\frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta \longrightarrow (1)$$

Putting $q=p$ in eqn (1), we get

$$\frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{p+1}{2}}{2\,\Gamma p+1} = \int_0^{\pi/2} (\sin\theta \cos\theta)^p \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} 2\sin\theta\cos\theta\right]^p d\theta = \frac{1}{2^p}\int_0^{\pi/2} (\sin 2\theta)^p \, d\theta$$

Put $2\theta = t \Rightarrow 2d\theta = dt$
when $\theta = 0$, $t=0$
$\theta = \frac{\pi}{2}$, $t = \pi$

$$= \frac{1}{2^p}\int_0^{\pi} \sin^p t \, \frac{dt}{2}.$$

$$= \frac{1}{2^p} \, 2\int_0^{\pi/2} \sin^p t \, \frac{dt}{2}$$

$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(a-x) = f(x)\right]$

$$= \frac{1}{2^p}\int_0^{\pi/2} \sin^p t \cos^0 t \, dt$$

$$\Rightarrow \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{p+1}{2}}{2\,\Gamma p+1} = \frac{1}{2^p} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{0+1}{2}}{2\,\Gamma\frac{p+0+2}{2}}$$

$$\Rightarrow \frac{\Gamma\frac{p+1}{2}}{\Gamma p+1} = \frac{1}{2^p} \frac{\sqrt{\pi}}{\Gamma\frac{p+2}{2}}$$

[Put $\frac{p+1}{2} = m \Rightarrow p = 2m-1$

$$\Rightarrow \frac{\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}} \frac{\sqrt{\pi}}{\Gamma\frac{2m+1}{2}} \longrightarrow (2)$$

$$\Rightarrow \boxed{\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m}$$

Multiplying eqn (2) by Γm both side, we get

$$\frac{\Gamma m \, \Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}} \frac{\Gamma\frac{1}{2} \, \Gamma m}{\Gamma m+\frac{1}{2}}$$

$$\Rightarrow \boxed{\beta(m,m) = 2^{1-2m} \beta(m, \tfrac{1}{2})}$$ Hence proved.

Ex-1: Prove that $\dfrac{\Gamma\frac{1}{3}\,\Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = 2^{1/3}\sqrt{\pi}$ (AKTU-2017)

Solⁿ: By Duplication formula, we have

$$\Gamma m\,\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m \longrightarrow (1)$$

Put $m = \frac{1}{3}$ in eqⁿ (1), we get

$$\Gamma\tfrac{1}{3}\,\Gamma\left(\tfrac{1}{3}+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2/3-1}}\,\Gamma\tfrac{2}{3}$$

$$\Rightarrow \boxed{\frac{\Gamma\frac{1}{3}\,\Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = \sqrt{\pi}\,2^{1/3}}$$ Hence proved.

Ex-2: Prove that

$$\beta(m,m)\times\beta\left(m+\tfrac{1}{2},\,m+\tfrac{1}{2}\right) = \frac{\pi}{m}\,2^{1-4m}$$ [AKTU-2008]

Solⁿ:

$$L.H.S = \beta(m,m)\times\beta\left(m+\tfrac{1}{2},\,m+\tfrac{1}{2}\right)$$

$$= \frac{\Gamma m\,\Gamma m}{\Gamma(m+m)} \times \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+m+\frac{1}{2}\right)} \qquad \left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right]$$

$$= \frac{(\Gamma m)^2 \times \left(\Gamma\left(m+\frac{1}{2}\right)\right)^2}{\Gamma 2m \times 2m\,\Gamma 2m} \qquad [\because \Gamma(2m+1) = 2m\,\Gamma 2m$$

$$= \left[\frac{\Gamma m\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma 2m}\right]^2 \cdot \frac{1}{2m}$$

$$= \left[\frac{\sqrt{\pi}}{2^{2m-1}}\right]^2 \cdot \frac{1}{2m}$$

[$\because$ By Duplication formula $\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m$]

$$= \frac{\pi}{2^{4m-2}} \cdot \frac{1}{2m}$$

$$= \frac{\pi}{m}\,2^{1-4m}$$

$$= R.H.S.$$

Engineering Mathematic-II KAS-203T Module-II * Multi. Cal.-II *

Lecture No- 18 [redacted]

Dirichlet's Integral→

Dirichlet's Integral is used to evaluate double and triple integral.

Dirichlet's Integral→ (For two dimension).

Let l and m are positive integer, Then

$$\boxed{\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1} h^{l+m}} \Rightarrow \boxed{\text{Only for first quadrant}}$$

where D is the domain $x \geq 0$, $y \geq 0$ and $x+y \leq h$.

Dirichlet's Integral→ (For three dimension).

If l, m, n are positive integer, Then

$$\boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n+1}} \Rightarrow \boxed{\text{Only for first Octant}}$$

where V is the domain or region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$.

Note→ i) ⓐ Area $= \iint_D dx\, dy$ ⓑ Mass $= \iint_D \rho\, dx\, dy$ where ρ density

ii) Volume $V = \iiint_V dx\, dy\, dz$

iii) Mass $= \iiint_V \rho\, dx\, dy\, dz$ where ρ is density.

Ex-1→ Find the area bound by the circle $x^2+y^2=a^2$ & $x \geq 0, y \geq 0$.

Solⁿ→ Given $x^2+y^2=a^2$ & $x \geq 0, y \geq 0$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \ \& \ x \geq 0, y \geq 0$$

Let $\frac{x^2}{a^2} = u \Rightarrow x = a u^{1/2}$ then $dx = a \frac{1}{2} u^{-1/2} du$

$\frac{y^2}{a^2} = v \Rightarrow y = a v^{1/2}$ then $dy = a \cdot \frac{1}{2} v^{-1/2} dv$

then $u+v \leq 1$, $u \geq 0, v \geq 0$.

Now required Area is

$$A = \iint_D dx\,dy \qquad \text{(Area in first quadrant)}$$

$$= \iint_D \frac{a}{2} u^{-1/2} \frac{a}{2} v^{-1/2}\, du\, dv$$

$$= \frac{a^2}{4} \iint_D u^{-1/2} v^{-1/2}\, du\, dv$$

$$= \frac{a^2}{4} \iint_D u^{\frac{1}{2}-1} v^{\frac{1}{2}-1}\, du\, dv$$

By Dirichlet's Integral $\left[\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma l\, \Gamma m}{\Gamma (l+m+1)} \right]$

$$= \frac{a^2}{4} \frac{\Gamma\frac{1}{2}\, \Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}$$

$$= \frac{a^2}{4} \frac{\sqrt{\pi}\,\sqrt{\pi}}{\Gamma 2} = \frac{a^2 \pi}{4} \qquad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\ \&\ \Gamma n = \lfloor n-1 \right]$$

Note→ The area of complete circle $= 4 \times$ Area in first quadrant

$$= 4 \times \frac{a^2 \pi}{4}$$

$$= a^2 \pi$$

Home Assignment

Ex-1→ Find the Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Hint→ [Find area in first quadrant by Dirichlet's Integral & then Required Area $= 4 \times$ Area in first quadrant $= \pi ab$.]

Ex-2→ Show that the area bounded by the curve $x^n + y^n = a^n$ & the coordinate axes in the first quadrant is $\frac{a^2 \left(\Gamma\frac{1}{n}\right)^2}{2n\, \Gamma\frac{2}{n}}$.

Ex-3→ Find the area & the mass contained in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1$ where $\alpha > 0$, $\beta > 0$ given that density at any point $P(x,y)$ is $k\sqrt{xy}$.

Ans $A = \frac{ab}{\alpha\beta} \frac{\Gamma\frac{1}{\alpha}\, \Gamma\frac{1}{\beta}}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+1\right)}$, $M = \frac{k(ab)^{3/2}}{\alpha\beta} \frac{\Gamma\frac{3}{2\alpha}\, \Gamma\frac{3}{2\beta}}{\Gamma\left(\frac{3}{2\alpha}+\frac{3}{2\beta}+1\right)}$

Ex-17 The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is $Kxyz$. [AKTU-2009, 2012, 2018].

Soln Given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $x\geq 0, y\geq 0, z\geq 0$.

Let $\frac{x}{a}=u \Rightarrow x=au$ then $dx=a\,du$

$\frac{y}{b}=v \Rightarrow y=bv$ then $dy=b\,dv$

$\frac{z}{c}=w \Rightarrow z=cw$ then $dz=c\,dw$

then $u+v+w\leq 1$, $u\geq 0, v\geq 0, w\geq 0$.

Now Volume $=\iiint_V dx\,dy\,dz$

$$=\iiint_V abc\;du\,dv\,dw$$

$$=abc\iiint_V u^0 v^0 w^0\,du\,dv\,dw$$

$$=abc\iiint_V u^{1-1} v^{1-1} w^{1-1}\,du\,dv\,dw$$

$\left[\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\right.$ for $x+y+z\leq 1, x\geq 0, y\geq 0, z\geq 0\Big]$

$$=abc\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)}$$

$$=abc\frac{\lfloor 0\,\lfloor 0\,\lfloor 0}{\lfloor 3} \qquad [\because \Gamma n=\lfloor n-1$$

$$\boxed{V=\frac{abc}{6}}.$$

& Mass $=\iiint_V \rho\,dx\,dy\,dz = \iiint_V Kxyz\,dx\,dy\,dz$

$$=K\iiint_V (au)(bv)(cw)\,abc\,du\,dv\,dw \qquad \text{[By Dirichlet's Integral]}$$

$$=Ka^2b^2c^2\iiint_V u^{2-1}v^{2-1}w^{2-1}\,du\,dv\,dw$$

$$=Ka^2b^2c^2\frac{\Gamma 2\,\Gamma 2\,\Gamma 2}{\Gamma(2+2+2+1)} = Ka^2b^2c^2\frac{\lfloor 1\lfloor 1\lfloor 1}{\lfloor 6} = \frac{Ka^2b^2c^2}{720}.$$

$$\Rightarrow \boxed{\text{Mass}=K\frac{a^2b^2c^2}{720}} \qquad \boxed{\&\ \lfloor n-1=\Gamma n}$$

Ex-2 Apply Dirichlet's Integral to find the volume and the mass contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, if the density at any point is $\rho(x,y,z)=kxyz$. [AKTU - 2006, 2007, 2011, 2013, 2015].

Sol^n Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $x\geq 0, y\geq 0, z\geq 0$ (for first octant)

Put $\frac{x^2}{a^2}=u \Rightarrow x=a\,u^{1/2}$ then $dx = a\cdot\frac{1}{2}u^{-1/2}du$

$\frac{y^2}{b^2}=v \Rightarrow y=b\,v^{1/2}$ then $dy = b\cdot\frac{1}{2}v^{-1/2}dv$

$\frac{z^2}{c^2}=w \Rightarrow z=c\,w^{1/2}$ then $dz = c\cdot\frac{1}{2}w^{-1/2}dw$

where $u+v+w\leq 1$, $u\geq 0, v\geq 0, w\geq 0$.

Required volume

$$V=\iiint_V dx\,dy\,dz = \iiint_V \frac{a}{2}u^{-1/2}\,\frac{b}{2}v^{-1/2}\,\frac{c}{2}w^{-1/2}\,du\,dv\,dw$$

$$= \frac{abc}{8}\iiint_V u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$$

$$= \frac{abc}{8}\iiint_V u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$$

$$= \frac{abc}{8}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)}$$

$\left[\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\right.$ for $x+y+z\leq 1$, $\left. x\geq 0, y\geq 0, z\geq 0\right\}$

$$= \frac{abc}{8}\,\frac{\sqrt{\pi}\,\sqrt{\pi}\,\sqrt{\pi}}{\Gamma\frac{5}{2}}$$

$\left[\because \Gamma\frac{1}{2}=\sqrt{\pi}\right]$.

$$= \frac{abc}{8}\,\frac{\pi\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$\left[\Gamma(n+1)=n\,\Gamma n\right]$

$$\Rightarrow \boxed{V=\frac{\pi abc}{6}}.$$

and required mass

$$M = \iiint_V \rho \, dx\,dy\,dz$$

$$= \iiint_V k\,xyz \, dx\,dy\,dz$$

$$= \iiint_V k (a u^{1/2})(b v^{\frac{1}{2}})(c w^{1/2}) \left(\frac{a}{2} u^{-1/2}\right)\left(\frac{b}{2} v^{-1/2}\right)\left(\frac{c}{2} w^{-1/2}\right) du\,dv\,dw$$

$$= k\frac{a^2b^2c^2}{8} \iiint_V u^0 \cdot v^0 \, w^0 \, du\,dv\,dw$$

$$= k\frac{a^2b^2c^2}{8} \iiint_V u^{1-1} v^{1-1} w^{1-1} du\,dv\,dw$$

$$= k\frac{a^2b^2c^2}{8} \frac{\Gamma 1 \, \Gamma 1 \, \Gamma 1}{\Gamma 1+1+1+1} \quad \text{[by Dirichlet's Integral]}$$

$$= k\frac{a^2b^2c^2}{8} \frac{\lfloor 0 \, \lfloor 0 \, \lfloor 0}{\lfloor 3} \quad [\because \Gamma n = \lfloor n-1 \,].$$

$$= k\frac{a^2b^2c^2}{48}$$

Ex-3: Find the volume of the solid by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \cdot \quad \text{(AKTU-2008)}$$

[Hint: Volume = 8 × Volume in first octant by Dirichlet Integral

$$= 8 \times \pi \frac{abc}{6} = \frac{4}{3}\pi abc \,.]$$

Ex-4: Find the volume contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. [AKTU-2014]

[Ans $\frac{\pi}{6}abc$]

Ex-5: Evaluate $\iiint xyz \, dx\,dy\,dz$ for all +ve value of variables of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

[Ans $\frac{a^2b^2c^2}{48}$].

Ex-6 Compute $\iiint_V x^2\,dx\,dy\,dz$ over volume of tetrahedron bounded by $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. [AKTU-2017]

[Ans $\frac{a^3bc}{60}$]

Ex-7 Evaluate $\iiint_V x^2yz\,dx\,dy\,dz$ throughout the volume bounded by planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. (AKTU-2007, 2015).

Solⁿ 7 Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $x\geq 0$, $y\geq 0$, $z\geq 0$.

Put $\frac{x}{a}=u \Rightarrow x=au$ then $dx=a\,du$

$\frac{y}{b}=v \Rightarrow y=bv$ then $dy=b\,dv$

$\frac{z}{c}=w \Rightarrow z=cw$ then $dz=c\,dw$

then $u+v+w\leq 1$, $u\geq 0$, $v\geq 0$, $z\geq 0$.

Now $\iiint_V x^2yz\,dx\,dy\,dz$

$$= \iiint_V (au)^2(bv)(cw)\,abc\,du\,dv\,dw$$

$$= a^3b^2c^2 \iiint_V u^2 v w\,du\,dv\,dw$$

$$= a^3b^2c^2 \iiint_V u^{3-1} v^{2-1} w^{2-1}\,du\,dv\,dw$$

$$= a^3b^2c^2 \frac{\Gamma 3\,\Gamma 2\,\Gamma 2}{\Gamma(3+2+2+1)}$$

$\left[\because \iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)} \text{ for } x+y+z\leq 1,\ x\geq 0, y\geq 0, z\geq 0\right]$

$$= a^3b^2c^2 \frac{\lfloor 2\,\lfloor 1\,\lfloor 1}{\lfloor 7}$$

$[\because \Gamma n = \lfloor n-1]$

$$= a^3b^2c^2 \frac{2\cdot 1\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \frac{a^3b^2c^2}{2520}.$$

Ex-8 Find the volume of the sphere $x^2+y^2+z^2=a^2$.

[AKTU-2019]

Sol^n Given $x^2+y^2+z^2=a^2$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}=1 \longrightarrow (1)$$

Let $\frac{x^2}{a^2}=u \Rightarrow x=a\,u^{1/2}$ then $dx=a\cdot\frac{1}{2}u^{-1/2}du$

$\frac{y^2}{a^2}=v \Rightarrow y=a\,v^{1/2}$ then $dy=a\frac{1}{2}v^{-1/2}dv$

$\frac{z^2}{a^2}=w \Rightarrow z=a\,w^{1/2}$ then $dz=a\cdot\frac{1}{2}w^{-1/2}dw$

& $u+v+w\le 1$, $u\ge 0$, $v\ge 0$, $w\ge 0$

[for first octant].

Now the Volume of the sphere

Volume of sphere in first octant by Dirichlet Integral

$$V = 8\times \text{Volume of sphere in first octant}$$

$$= 8\iiint_V dx\,dy\,dz.$$

$$= 8\iiint_V \left(\frac{a}{2}u^{-1/2}\right)\left(\frac{a}{2}v^{-1/2}\right)\left(\frac{a}{2}w^{-1/2}\right)du\,dv\,dw$$

$$= 8\,\frac{a^3}{8}\iiint_V u^{-1/2}\,v^{-1/2}\,w^{-1/2}\,du\,dv\,dw$$

$$= a^3\iiint_V u^{\frac{1}{2}-1}\,v^{\frac{1}{2}-1}\,w^{\frac{1}{2}-1}\,du\,dv\,dw$$

$$\left[\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)},\ \text{for } x+y+z\le 1,\ x\ge 0, y\ge 0, z\ge 0\right]$$

$$= a^3\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)}$$

$$= a^3\,\frac{\sqrt{\pi}\,\sqrt{\pi}\,\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$\boxed{V=\frac{4}{3}a^3\pi}$$

Ex-9 → Find the volume of the solid surrounded by the surface $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$. (AKTU-2009)

Hint → [V = 8 x Volume in first octant] [Ans - $\frac{4\pi abc}{35}$]

Ex-10 → Find the volume of the solid bounded by the coordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$.

[Ans - $\frac{abc}{90}$].

Ex-11 → Evaluate $I=\iiint_V x^{\alpha-1}y^{\beta-1}z^{\gamma-1}\,dx\,dy\,dz$ where V is the region in the first octant bounded by sphere $x^2+y^2+z^2=1$ and the coordinate planes. (AKTU-2004)

Ans → $I=\frac{1}{8}\frac{\Gamma\frac{\alpha}{2}\,\Gamma\frac{\beta}{2}\,\Gamma\frac{\gamma}{2}}{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+1\right)}$.

Ex-12 → Evaluate the integral $\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$ where x, y, z are all +ve but limited by the Condn $\left(\frac{x}{a}\right)^{p}+\left(\frac{y}{b}\right)^{q}+\left(\frac{z}{c}\right)^{r}\le 1$ [AKTU-2011]

Ans → $\frac{a^l b^m c^n}{pqr}\frac{\Gamma\frac{l}{p}\,\Gamma\frac{m}{q}\,\Gamma\frac{n}{r}}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$.

Ex-13 → Find the volume of the solid bdd by coordinate planes and the surface

$\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}+\left(\frac{z}{c}\right)^{2n}=1$, n being a +ve integer.

Ans → $\frac{abc}{12n^2}\frac{\left(\Gamma\frac{1}{2n}\right)^3}{\Gamma\frac{3}{2n}}$.

Ex-14: Find the mass of a solid $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$, the density at any point being $\rho = k x^{l-1} y^{m-1} z^{n-1}$ where x, y, z are all +ve. [AKTU-2003, 2016].

Sol: Given $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$

Put $\left(\frac{x}{a}\right)^p = u \Rightarrow x = a u^{1/p}$ then $dx = a \frac{1}{p} u^{\frac{1}{p}-1} du$

$\left(\frac{y}{b}\right)^q = v \Rightarrow y = b v^{1/q}$ then $dy = b \cdot \frac{1}{q} v^{\frac{1}{q}-1} dv$

$\left(\frac{z}{c}\right)^r = w \Rightarrow z = c w^{1/r}$ then $dz = c \cdot \frac{1}{r} w^{\frac{1}{r}-1} dw$

& $u+v+w \le 1$, $u \ge 0, v \ge 0, w \ge 0$.

Now $M = \iiint_V \rho \, dx\, dy\, dz$

$= \iiint_V k x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$

$= k \iiint_V (a u^{1/p})^{l-1} (b v^{1/q})^{m-1} (c w^{1/r})^{n-1} \frac{a}{p} u^{\frac{1}{p}-1} \frac{b}{q} v^{\frac{1}{q}-1} \frac{c}{r} w^{\frac{1}{r}-1} du\, dv\, dw$

$= k \frac{a^l b^m c^n}{p q r} \iiint_V u^{\frac{l}{p}-1} v^{\frac{m}{q}-1} w^{\frac{n}{r}-1} du\, dv\, dw$

$= k \frac{a^l b^m c^n}{p \cdot q \cdot r} \frac{\Gamma\frac{l}{p} \Gamma\frac{m}{q} \Gamma\frac{n}{r}}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$

$\left[\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)} \text{ for } x+y+z \le 1, x \ge 0, y \ge 0, z \ge 0 \right]$.

Ex-15: Find the mass of the region bdd by ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, if the density varies as the square of the distance from the centre. [AKTU-2010, 2013]

[Hint $\rho = k(x^2+y^2+z^2)$]

Ans → $\frac{k \pi abc}{16}(a^2+b^2+c^2)$.

Ex-16: Evaluate $\iiint_V (ax^2+by^2+cz^2)\,dx\,dy\,dz$ where V is the region bound by $x^2+y^2+z^2 \le 1$ [AKTU-2014].

Sol: Given $x^2+y^2+z^2 \le 1$.

Let $x^2 = u \Rightarrow x = u^{1/2}$ then $dx = \frac{1}{2}u^{-1/2}du$

$y^2 = v \Rightarrow y = v^{1/2}$ then $dy = \frac{1}{2}v^{-1/2}dv$

$z^2 = w \Rightarrow z = w^{1/2}$ then $dz = \frac{1}{2}w^{-1/2}dw$ & $u+v+w \le 1,\ u\ge 0, v\ge 0, w\ge 0$

Now $\iiint_V (ax^2+by^2+cz^2)\,dx\,dy\,dz$

$$= a\iiint_V x^2\,dx\,dy\,dz + b\iiint_V y^2\,dx\,dy\,dz + c\iiint_V z^2\,dx\,dy\,dz$$

$$= a\iiint_V u\,\tfrac{1}{2}u^{-1/2}\,\tfrac{1}{2}v^{-1/2}\,\tfrac{1}{2}w^{-1/2}\,du\,dv\,dw$$
$$+ b\iiint_V v\,\tfrac{1}{2}u^{-1/2}\,\tfrac{v^{-1/2}}{2}\,\tfrac{w^{-1/2}}{2}\,du\,dv\,dw$$
$$+ c\iiint_V w\left(\tfrac{1}{2}u^{-1/2}\right)\left(\tfrac{1}{2}v^{-1/2}\right)\left(\tfrac{1}{2}w^{-1/2}\right)du\,dv\,dw$$

$$= \frac{a}{8}\iiint_V u^{\frac{3}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$$
$$+ \frac{b}{8}\iiint_V u^{\frac{1}{2}-1}v^{\frac{3}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$$
$$+ \frac{c}{8}\iiint_V u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{3}{2}-1}\,du\,dv\,dw$$

$$= \frac{a}{8}\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} + \frac{b}{8}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{3}{2}+\frac{1}{2}+1\right)} + \frac{c}{8}\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{3}{2}+1\right)}$$

$$\left[\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)} \text{ for } x+y+z\le 1 \text{ \& } x\ge 0, y\ge 0, z\ge 0\right]$$

$$= \frac{1}{8}(a+b+c)\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{5}{2}+1\right)}$$

$$= \frac{1}{8}(a+b+c)\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\frac{3}{2}} \qquad \left[\because \Gamma n = \lfloor n-1,\quad \Gamma(n+1) = n\Gamma n\right]$$

$$= \frac{1}{8}(a+b+c)\frac{4\pi}{15}$$

$$= \frac{\pi(a+b+c)}{30}.$$

* Lecture - No - 19 *

LIOUVILLE'S EXTENSION OF DIRICHLET INTEGRAL →

If the variables x, y, z are all positive such that $h_1 \le x+y+z \le h_2$ then

$$\boxed{\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n} \int_{h_1}^{h_2} f(h)\, h^{l+m+n-1}\, dh}$$

Ex-1→ Evaluate $\iiint e^{x+y+z}\, dx\,dy\,dz$, taken over positive octant such that $x+y+z \le 1$.

Sol^n By LIOUVILLE's Extension of dirichlet integral we have

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n} \int_{h_1}^{h_2} f(h)\, h^{l+m+n-1}\, dh \longrightarrow (1)$$

for $h_1 \le x+y+z \le h_2$.

Given $x+y+z \le 1$

or $0 \le x+y+z \le 1$.

Then $\iiint e^{x+y+z}\, dx\,dy\,dz = \iiint e^{x+y+z} x^{1-1} y^{1-1} z^{1-1}\, dx\,dy\,dz$

$$= \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma 1+1+1} \int_0^1 e^h\, h^{1+1+1-1}\, dh.$$

$$= \frac{1}{\underline{|2}} \int_0^1 h^2 e^h\, dh$$

$$= \frac{1}{2}\left[h^2 e^h - \int 2h\, e^h\, dh\right]_0^1 = \left(\frac{1}{2} h^2 e^h\right)_0^1 - \left[h e^h - e^h\right]_0^1 = \frac{1}{2} e - e^0$$

$$= \frac{1}{2} e - 1.$$

Ex-2) Evaluate $\iiint \log(x+y+z)\,dx\,dy\,dz$, the integral extending over all +ve and zero values of x, y, z subject to $x+y+z \le 1$. [Ans - $-\frac{1}{18}$].

Ex-3) Evaluate $\iiint_V e^{-(x+y+z)}\,dx\,dy\,dz$, where the region of integration is bounded by planes $x=0, y=0, z=0$ and $x+y+z=a$, $a>0$. [AKTU-2008]

$$\text{Ans}\quad 1 - e^{-a}\left[1 + a + \frac{a^2}{2}\right].$$

Ex-4) Evaluate $\iiint xyz \sin(x+y+z)\,dx\,dy\,dz$, the integral being extended to all +ve values of the variables subject to the cond^n $x+y+z \le \frac{\pi}{2}$.

[AKTU-2011]

Ans $\to$ $\frac{1}{\lfloor 5}\left[\frac{5}{16}\pi^4 - 15\pi^2 + 120\right]$.

Ex-5) Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$, the integral being taken throughout the valme bdd by planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. [Ans]

Ex-6) Evaluate $\iiint (x+y+z)^9\,dx\,dy\,dz$ over the region bounded by $x \ge 0, y \ge 0, z \ge 0$ and $x+y+z \le 1$.

[Ans - $\frac{1}{24}$].

Ex-7) Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2} = \frac{3}{4} - \log 2$, the integral being taken throughout the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$. [AKTU-2008]

Sol^n) Here $0 \le x+y+z \le 1$.

Now $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2}$

$$= \iiint \frac{1}{[(x+y+z)+1]^2} \cdot x^{1-1} y^{1-1} z^{1-1} \, dx\,dy\,dz$$

$$= \frac{\Gamma 1 \, \Gamma 1 \, \Gamma 1}{\Gamma 1+1+1} \int_0^1 \frac{1}{(h+1)^2} \cdot h^{1+1+1-1} \, dh$$

$$\left[\because \iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1} \, dx\,dy\,dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n} \int_{h_1}^{h_2} f(h)\, h^{l+m+n-1} \, dh \right]$$

$$= \frac{1}{\lfloor 2} \int_0^1 \frac{h^2}{(h+1)^2} \, dh.$$

$$= \frac{1}{2} \int_1^2 \frac{(t-1)^2}{t^2} \, dt.$$

Let $h+1 = t$
$\Rightarrow h = t-1$
$[\because dh = dt]$
when $h=0$, $t=1$
$h=1$, $t=2$.

$$= \frac{1}{2} \int_1^2 \left[1 - \frac{2}{t} + \frac{1}{t^2}\right] dt$$

$$= \frac{1}{2} \left[t - 2\log t - \frac{1}{t} \right]_1^2$$

$$= \frac{1}{2} \left[(2-1) - 2(\log 2 - \log 1) - \left(\frac{1}{2} - 1\right) \right]$$

$$= \frac{1}{2} \left[1 - 2\log 2 + \frac{1}{2} \right] = \frac{3}{4} - \log 2. \text{ [Ans].}$$

V.V.Q

Ex-8 (i) Evaluate the integral $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ the integral being extended to all +ve values of the variables for which the expression is real. [AKTU-2010, 2012]

(ii) Prove that $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$, the integral being extended to all +ve values of the variables for which the expression is real. [AKTU - 2014, 2016, 2013].

Solⁿ (ii) The expression $\frac{1}{\sqrt{1-x^2-y^2-z^2}}$ will be real if $1-x^2-y^2-z^2>0$

Hence the given integral is extended for all +ve values of the variables x, y, z such that $0<x^2+y^2+z^2<1$

Let $x^2=u \Rightarrow x=u^{1/2}$ then $dx=\frac{1}{2}u^{-1/2}du$

$y^2=v \Rightarrow y=v^{1/2}$ then $dy=\frac{1}{2}v^{-1/2}dv$

$z^2=w \Rightarrow z=w^{1/2}$ then $dz=\frac{1}{2}w^{-1/2}dw$

then $0<u+v+w<1$.

Hence $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$

$$=\iiint \frac{\frac{1}{2}u^{-1/2}\,\frac{1}{2}v^{-1/2}\,\frac{1}{2}w^{-1/2}\,du\,dv\,dw}{\sqrt{1-(u+v+w)}}$$

$$=\frac{1}{8}\iiint \frac{1}{\sqrt{1-(u+v+w)}}\,u^{\frac{1}{2}-1}\,v^{\frac{1}{2}-1}\,w^{\frac{1}{2}-1}\,du\,dv\,dw$$

$$=\frac{1}{8}\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}\int_0^1 \frac{1}{\sqrt{1-h}}\,h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,dh$$

$\left[\because \iiint f(x+y+z)\,x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(h)\,h^{l+m+n-1}\,dh\right.$ for $\left.h_1\le x+y+z\le h_2\right]$

$$=\frac{1}{8}\,\frac{\sqrt{\pi}\,\sqrt{\pi}\,\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}}\int_0^1 h^{\frac{3}{2}-1}(1-h)^{\frac{1}{2}-1}\,dh$$

$$=\frac{\pi}{4}\,\beta\left(\frac{3}{2},\frac{1}{2}\right)$$

$\left[\because \int_0^1 x^{m-1}(1-x)^{n-1}dx=\beta(m,n)\right.$ & $\left.\beta(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}\right]$.

$$=\frac{\pi}{4}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)}$$

$$=\frac{\pi}{4}\,\frac{\frac{1}{2}\sqrt{\pi}\,\sqrt{\pi}}{\Gamma 2}$$

$\left[\because \Gamma(n+1)=n\,\Gamma n \;\&\; \Gamma\frac{1}{2}=\sqrt{\pi}\right]$.

$$=\frac{1}{8}\pi^2.$$

MATHEMATICS-I
KAS-103T
Lecture No - 34

* Module-IV *

* Multivariable Calculus *

By. Dr. Anuj Kumar

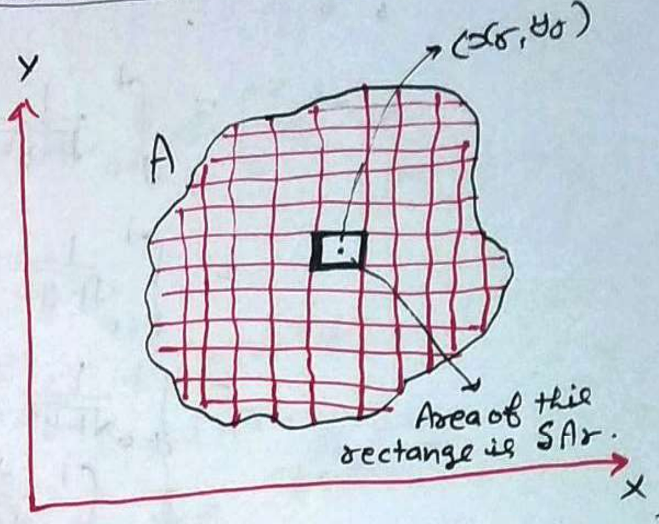
Double Integral (for Cartesian Coordinates) →

Let $\iint_A f(x,y)\, dA$

Let us consider a func^n $f(x,y)$ of two variables x & y defined in the finite region A of x-y plane.

Divide the region A into elementary area

$\delta A_1, \delta A_2, \delta A_3, \ldots, \delta A_n$

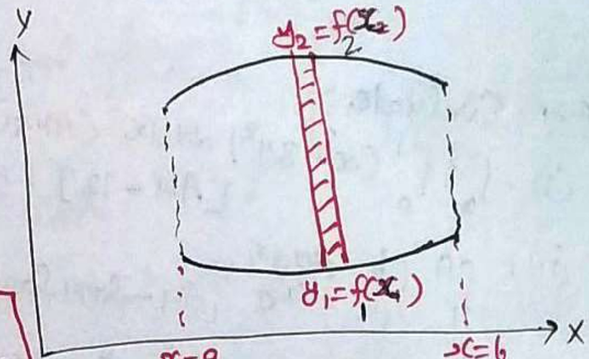


Then $\iint_A f(x,y)\, dA = \lim_{\substack{n\to\infty \\ \delta A \to 0}} \left[f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \cdots + f(x_n,y_n)\,\delta A_n \right]$

Evaluation of the double integral →

Method-I If A is described as

$a \le x \le b$ and $y_1 \le y \le y_2$ or $f_1(x) \le y \le f_2(x)$



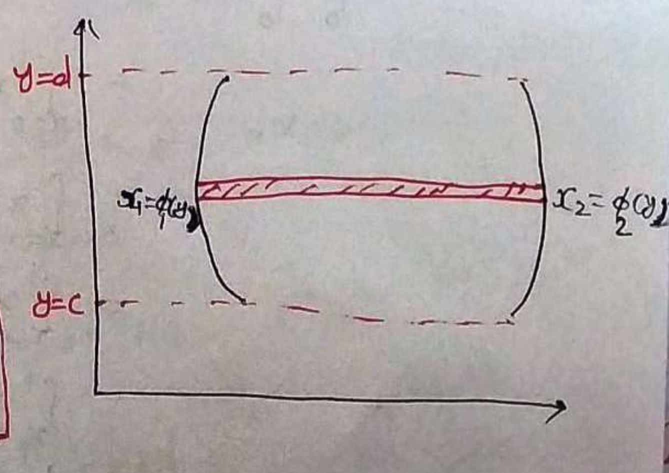
Then

$$\iint_A f(x,y)\, dA = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$$

Method-II →

If A is described as

$x_1 \le x \le x_2$ and $c \le y \le d$
or
$x_1 = \phi_1(y) \le x \le x_2 = \phi_2(y)$



$$\iint_A f(x,y)\, dA = \int_{y=c}^{d} \int_{x=\phi_1(y)}^{\phi_2(y)} f(x,y)\, dx\, dy.$$

Ex-1 Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$ (AKTU-2016). [Type-1 If limit is given]

Sol^n Given $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_{y=0}^1 \int_{x=0}^1 \frac{dx\ \ dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$$= \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} \left[\int_{x=0}^1 \frac{dx}{\sqrt{1-x^2}} \right] dy$$

$$= \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} \left[\sin^{-1} x \right]_0^1 dy$$

$$= \int_{y=0}^1 \frac{1}{\sqrt{1-y^2}} [\sin^{-1} 1 - \sin^{-1} 0]\, dy$$

$$= \frac{\pi}{2} \int_{y=0}^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2} \left[\sin^{-1} y \right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1} 1 - \sin^{-1} 0 \right]$$

$$= \frac{\pi}{2} \times \frac{\pi}{2} = \frac{\pi^2}{4}.$$

Ex-2 Evaluate

i) $\int_0^3 \int_0^1 (x^2 + 3y^2)\, dy\, dx$ (AKTU-2019) [Ans - 12]

ii) $\int_0^3 \int_1^2 xy\,(1+x+y)\, dy\, dx$ [Ans $30\frac{3}{4}$].

iii) $\int_1^a \int_1^b \frac{dy\,dx}{xy}$ [Ans - $\log a \log b$].

iv) $\int_0^1 \int_1^2 xy\, dy\, dx$. [Ans - $\frac{3}{4}$].

Ex-3 Evaluate $\int_0^1 \int_0^{x^2} x e^y\, dy\, dx$. (AKTU-2018).

Sol^n Given $\int_0^1 \int_0^{x^2} x e^y\, dy\, dx = \int_{x=0}^1 \int_{y=0}^{x^2} x e^y\, dy\, dx$.

$$= \int_{x=0}^1 x \left[\int_{y=0}^{x^2} e^y\, dy \right] dx$$

$$= \int_{x=0}^1 x \left[e^y \right]_0^{x^2} dx.$$

$$= \int_{x=0}^1 x \left[e^{x^2} - e^0 \right] dx.$$

$$= \int_{x=0}^1 x e^{x^2} dx - \int_{x=0}^1 x\, dx$$

$$= \frac{1}{2}\left[e^{x^2} \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$$ [Let $x^2 = t$, $2x\,dx = dt$]

$$= \frac{1}{2}[e^1 - e^0] - [\tfrac{1}{2} - 0] = \tfrac{1}{2}e - 1$$

Ex-4: (i) Evaluate $\int_0^a \int_0^x xy\, dy\, dx$ (AKTU-2014)

[Ans $-\frac{1}{8}$]

(ii) Evaluate $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dx\, dy$ [Ans $-$ $8\log 8 - 16 + e$.

Ex-5: Evaluate

(i) $\int_1^2 \int_0^x \frac{dy\, dx}{x^2+y^2}$ (AKTU-2007) [Ans $\frac{\pi}{4}\log 2$]

(ii) $\int_0^1 dx \int_0^x e^{y/x}\, dy$ [Ans $-\frac{1}{2}(e-1)$

(iii) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$ [Ans $\frac{1}{2}$]

Ex-6: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$. (AKTU-2009).

Ans: Given $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2} = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2} \right] dx.$$

$$= \int_{x=0}^{1} \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx. \qquad \left[\because \int \frac{dy}{a^2+y^2} = \frac{1}{a}\tan^{-1}\frac{y}{a}\right.$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \tan^{-1} 0 \right] dx$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0)\, dx \qquad \left[\because \int \frac{dx}{\sqrt{1+x^2}} = \log[x+\sqrt{1+x^2}]\right].$$

$$= \frac{\pi}{4} \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4} \left[\log(x+\sqrt{1+x^2}) \right]_0^1 = \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log 1\right]$$

$$= \frac{\pi}{2} \log(1+\sqrt{2}).$$

Ex-7: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$.

Sol^n: Given $\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

$$= \int_{y=0}^{a} \left[\int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx \right] dy.$$

$$= \int_{y=0}^{a} \left[\frac{1}{2} x \sqrt{(a^2-y^2)-x^2} + \frac{1}{2}(a^2-y^2) \sin^{-1} \frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy \qquad \left[\because \int \sqrt{a^2-x^2} = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]$$

$$= \int_{y=0}^{a} \left[0 + \frac{1}{2}(a^2-y^2) \sin^{-1}\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}} \right] dy.$$

$$= \int_{y=0}^{a} \frac{1}{2}(a^2-y^2) \sin^{-1} 1 \, dy.$$

$$= \frac{\pi}{2} \times \frac{1}{2} \int_{y=0}^{a} (a^2-y^2) = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3} - 0\right] = \frac{\pi}{4} \times \frac{2a^3}{3} = \frac{\pi a^3}{6}.$$

$\boxed{\text{Type-II} \to \text{If limit is not given} \to}$

Ex-1: Evaluate $\iint xy(x+y)\,dx\,dy$ over the area b/w $y=x^2$ & $y=x$.
(AKTU-2011, 2012, 2014)

Sol^n: Given curve are

$y = x^2$

& $y = x$

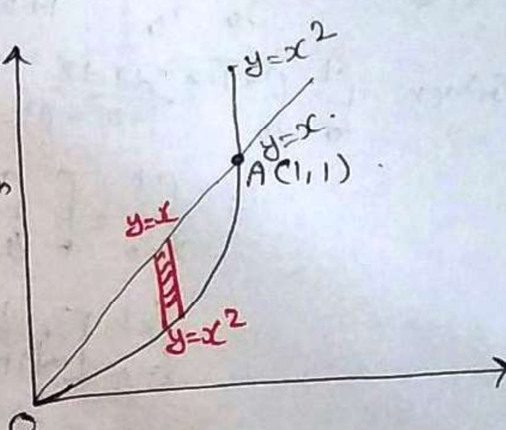
On solving, we get point of intersection

$x = x^2 \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0,\ x = 1$

Then $y = 0,\ y = 1$.



Hence points are $O(0,0)$, $A(1,1)$.

Now for the area b/w $y=x^2$ & $y=x$, we have

$0 \le x \le 1$, $x^2 \le y \le x$.

Then $$\iint xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{x} [x^2 y + x y^2]\,dy\right]dx.$$

$$= \int_{x=0}^{1}\left[x^2 \frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{x} dx.$$

$$= \int_{x=0}^{1}\left[x^2\cdot\frac{x^2}{2} + x\cdot\frac{x^3}{3} - x^2\cdot\frac{x^4}{2} - x\cdot\frac{x^6}{3}\right]dx.$$

$$= \int_{x=0}^{1}\left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right]dx.$$

$$= \int_{x=0}^{1}\left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right]dx = \left[\frac{5}{6}\cdot\frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}.$$ Ans

Ex-2: Evaluate $\iint_R x^2\,dx\,dy$, R is the two dimensional region bdd by the curves $y = x$ & $y = x^2$. $\left[\text{Ans} - \frac{1}{20}\right]$

Ex-3: Evaluate $\iint_D (x^2+y^2)\,dx\,dy$ where D is bdd by $y=x$ & $y^2=4x$. $\left[\text{Ans } \frac{768}{35}\right]$

Ex-4: Evaluate $\iint_R y\,dx\,dy$ where R is the region bdd by the parabolas $y^2=4x$ & $x^2=4y$. $\left[\text{Ans} - \frac{48}{5}\right]$. (AKTU-2010)

Ex-5: Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate $x=2a$ & the curve $x^2=4ay$. (AKTU-2012).

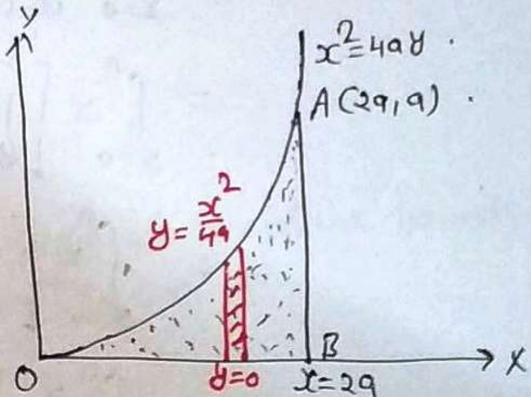
Sol^n: Given region is bdd by x-axis, ordinate $x=2a$ & $x^2=4ay$. i.e region is OBAO.

Put $x=2a$ in $x^2=4ay$

we get $4a^2 = 4ay$

$\Rightarrow y = a$.

$\therefore$ point A is $(2a, a)$



Hence $0 \le x \le 2a$ & $0 \le y \le \frac{x^2}{4a}$.

Now $\iint_A xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx$.

$$= \int_{x=0}^{2a} x\left[\int_{y=0}^{x^2/4a} y\,dy\right]dx$$

$$= \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx.$$

$$= \int_{x=0}^{2a} x\left[\frac{1}{2}\frac{x^4}{16a^2} - 0\right]dx$$

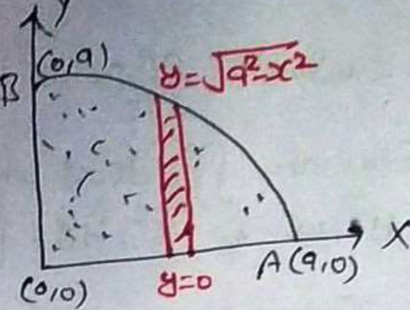
$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{3a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\left[\frac{(2a)^6}{6} - 0\right]$$

$$= \frac{a^4}{3}. \text{ Ans}.$$

Ex-6: Evaluate $\iint_R xy\,dx\,dy$, where R is the +ve quadrant of the circle $x^2+y^2=a^2$.

Sol^n:



In +ve quadrant of the circle $x^2+y^2=a^2$

$0 \le x \le a$ & $0 \le y \le \sqrt{a^2-x^2}$

Hence
$$\iint_R xy\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} x\,y\,dy\,dx.$$

$$= \int_{x=0}^{a} x\left[\int_{y=0}^{\sqrt{a^2-x^2}} y\,dy\right]dx$$

$$= \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} x\left[\frac{a^2-x^2}{2}-0\right]dx.$$

$$= \frac{1}{2}\int_{x=0}^{a}(a^2x-x^3)\,dx.$$

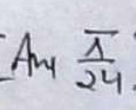
$$= \frac{1}{2}\left[a^2\frac{x^2}{2}-\frac{x^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}-0\right]$$

$$= \frac{1}{2}\times a^4\times\frac{1}{4} = \frac{a^4}{8}.$$

Ex-7: Evaluate $\iint x^2y^2\,dx\,dy$ over the circle $x^2+y^2=1$. [Ans $\frac{\pi}{24}$]

Ex-8: Evaluate $\iint xy\,dx\,dy$ over the circle $x^2+y^2=1$.

Ex-9: Evaluate $\iint xy\,dx\,dy$ over the region in the +ve quadrant for which $x+y\le 1$. [Ans $\frac{1}{24}$]



Ex-10: Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the +ve quadrant for which $x+y\le 1$. [Ans $\frac{1}{6}$].

Ex-11: Evaluate $\iint (x^2+y^2)\,dx\,dy$ throughout the area enclosed by the curve $y=4x$, $x+y=3$, $y=0$ and $y=2$.
(AKTU-2012).

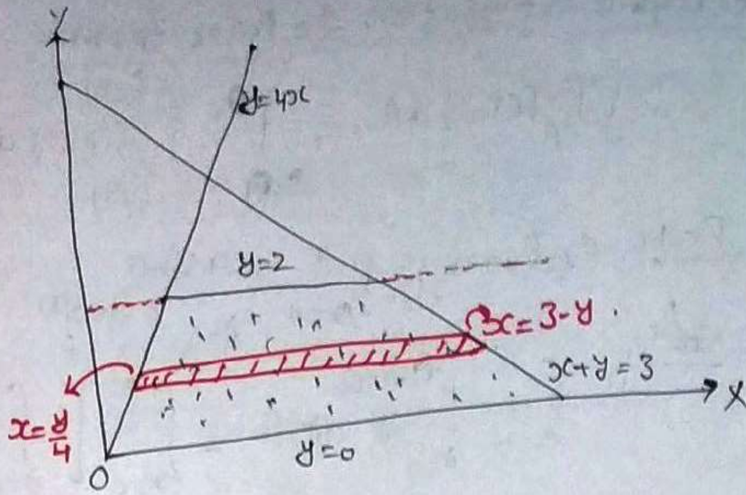
Sol^n 11:

Given Area enclosed by the curve

$y = 4x$

$x + y = 3$

$y = 0$ & $y = 2$.



Then

$$\iint_A (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{2} \int_{x=\frac{y}{4}}^{3-y} (x^2+y^2)\,dx\,dy$$

$$= \frac{463}{48} \quad \text{(Solve by yourself)}.$$

Ex-12: Evaluate $\iint y\,dx\,dy$ over the part of the plane bounded by the line $y=x$ and the parabola $y = 4x - x^2$.

Sol^n: Given $y = 4x - x^2$

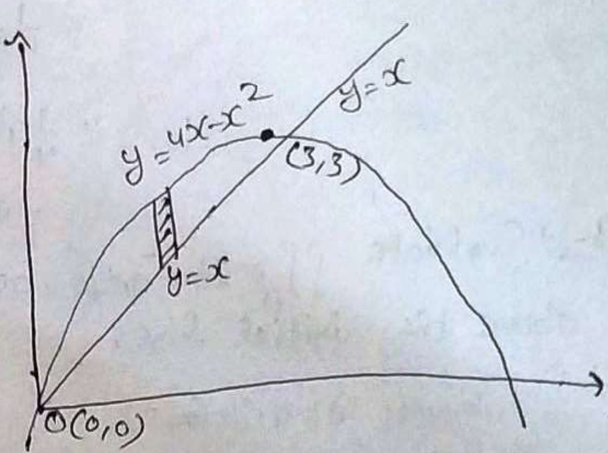
$\Rightarrow y = -(x^2 - 4x)$

$\Rightarrow y = -(x^2 - 4x + 4 - 4)$

$\Rightarrow y - 4 = -(x-2)^2$

$\Rightarrow X^2 = -Y$

i.e. a parabola with vertex $(2, 4)$



On solving $y = x$ & $y = 4x - x^2$

$x = 4x - x^2 \Rightarrow x^2 - 3x = 0$

$\Rightarrow x = 0, 3$ then $y = 0, 3$.

$\therefore$ Intersecting points are $(0,0), (3,3)$.

Hence $$\iint_R y\,dx\,dy = \int_{x=0}^{3} \int_{y=x}^{4x-x^2} y\,dy\,dx = \frac{54}{5} \text{ (try yourself)}$$

Ex-13: Evaluate $\iint_S \sqrt{xy - y^2}\,dx\,dy$, where S is a triangle with vertices $(0,0)$, $(10,1)$ & $(1,1)$.

Double integration in Polar form →

$$\iint_A f(x,y)\,dA = \int_{\theta=\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\,dr\,d\theta.$$

Ex-1: Evaluate; (i) $\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta$. (ii) $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2\sin\theta\,dr\,d\theta = \frac{4}{3}a^3$.

Solⁿ: (i)

$$\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta}d\theta$$

$$= \frac{1}{2}a^2\int_{\theta=0}^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{1}{4}a^2\int_{\theta=0}^{\pi}2\sin^2\theta\,d\theta = \frac{1}{4}a^2\int_0^{\pi}(1-\cos2\theta)\,d\theta$$

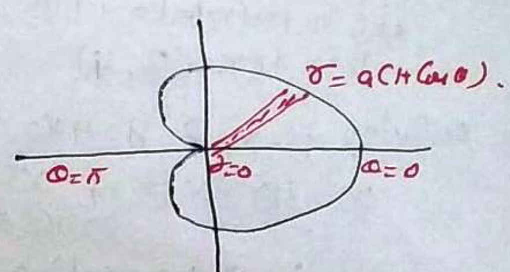
$$= \frac{1}{4}a^2\left[\theta - \frac{\sin2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}a^2\left[\pi - \frac{\sin2\pi}{2} - 0\right]$$

$$= \frac{1}{4}a^2\pi.$$

Ex-2: Evaluate $\iint_A r\sin\theta\,dr\,d\theta$, over area of cardiods $r=a(1+\cos\theta)$ above the initial line.

Solⁿ: Given $r = a(1+\cos\theta)$



Put $r=0$, $a(1+\cos\theta)=0$

$\Rightarrow \cos\theta = -1 = \cos\pi$

$\Rightarrow \boxed{\theta = \pi}$

Now

$$\iint_A r\sin\theta\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\sin\theta\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)}d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi}\sin\theta\cdot a^2(1+\cos\theta)^2\,d\theta$$

$$= -\frac{a^2}{2}\left[\frac{(1+\cos\theta)^3}{3}\right]_0^{\pi} = \frac{4a^2}{3}. \quad \left[\begin{array}{l}\text{Let } 1+\cos\theta = t\\ -\sin\theta\,d\theta = dt\end{array}\right]$$

MATHEMATICS-I
KAS-103T
Lecture-No-35

Module-IV
By. Dr. Anuj Kumar

Triple Integral→
Let $f(x,y,z)$ be a continuous funcⁿ at every point of finite region V of three dimensional space. Then Triple Integral is

$$\boxed{\iiint_V f(x,y,z)\,dV = \int_{x=a}^{b}\int_{y=\phi_1(x)}^{\phi_2(x)}\int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dy\,dx.}$$

where V is the region bounded by the curves
$a \le x \le b$, $\phi_1(x) \le y \le \phi_2(x)$, $f_1(x,y) \le z \le f_2(x,y)$.

Ex-1→ Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx$. (AKTU-2010).

Solⁿ→ Let $\displaystyle I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx$.

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\,dz\,dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx.$$

$$\left[\because \int\frac{dz}{\sqrt{a^2-z^2}} = \sin^{-1}\frac{z}{a}\right]$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left[\sin^{-1}1 - \sin^{-1}0\right] dy\,dx.$$

$$= \frac{\pi}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\,dx.$$

$$= \frac{\pi}{2}\int_{x=0}^{1}\left[y\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_{x=0}^{1}\sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{2}\,\frac{\sin^{-1}1}{2} = \frac{\pi}{2}\cdot\frac{\pi}{4} = \frac{\pi^2}{8}.$$

Ex-2: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$ (AKTU-2017).

Sol^n: Given

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xy\, z \, dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy[(1-x^2)-y^2]\, dy\, dx.$$

$$= \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} [(x-x^3)y - xy^3]\, dy\, dx.$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[(x-x^3)\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} - x\left(\frac{y^4}{4}\right)_0^{\sqrt{1-x^2}}\right] dx.$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[(x-x^3)\frac{(1-x^2)}{2} - x\frac{(1-x^2)^2}{4}\right] dx.$$

$$= \frac{1}{2}\int_{x=0}^{1} x\,\frac{(1-x^2)^2}{4}\, dx.$$

$$= \frac{1}{8}\int_{x=0}^{1} x(1+x^4-2x^2)\, dx$$

$$= \frac{1}{8}\int_0^1 (x+x^5-2x^3)\, dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2}+\frac{x^6}{6}-2\frac{x^4}{4}\right]_0^1 = \frac{1}{8}\left[\frac{1}{2}+\frac{1}{6}-\frac{1}{2}\right] = \frac{1}{48}.$$

Ex-3: Evaluate $\int_0^1 \int_0^2 \int_0^2 xy \, dx\, dy\, dz$. [Ans - 3].

Ex-4: Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} \, dx\, dy\, dz$. [Ans - $(e-1)^3$].

Ex-5: Evaluate $\iiint_R (x+y+z)\, dx\, dy\, dz$, where $R: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$.

(AKTU-2014, 2016, 2018).

Ex-6: Evaluate $\iiint_R (x-2y+z)\,dz\,dy\,dx$, where R is the region determined by $0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le x+y$.

(AKTU - 2006, 2008)

Solⁿ: Given region is

$0 \le x \le 1, \quad 0 \le y \le x^2, \quad 0 \le z \le x+y.$

Then $\iiint_R (x-2y+z)\,dz\,dy\,dx$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} \int_{z=0}^{x+y} [(x-2y)+z]\,dz\,dy\,dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} \left[(x-2y)\,z + \frac{z^2}{2}\right]_0^{x+y} dy\,dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} \left[(x-2y)(x+y) + \frac{(x+y)^2}{2}\right] dy\,dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} \left[(x^2+xy-2xy-2y^2) + \frac{(x^2+y^2+2xy)}{2}\right] dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{x^2} \frac{(2x^2-2xy-4y^2+x^2+y^2+2xy)}{2}\,dy\,dx.$$

$$= \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{x^2} (3x^2-3y^2)\,dy\,dx.$$

$$= \frac{3}{2}\int_{x=0}^{1} \int_{y=0}^{x^2} (x^2-y^2)\,dy\,dx$$

$$= \frac{3}{2}\int_{x=0}^{1} \left[x^2y - \frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \frac{3}{2}\int_{x=0}^{1} \left[x^4 - \frac{x^6}{3}\right] dx$$

$$= \frac{3}{2}\left[\frac{x^5}{5} - \frac{x^7}{21}\right]_0^1 = \frac{3}{2}\left[\frac{1}{5} - \frac{1}{21}\right]$$

$$= \frac{3}{2} \times \frac{21-5}{21\times 5} = \frac{8}{35}.$$

Ex-7: Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$. [Ans - 8π]

Ex-8: Evaluate $\int_0^a \int_0^a \int_0^a (yz + zx + xy)\, dx\, dy\, dz$.
[Ans - $\frac{3}{4}a^5$]

Ex-9: Evaluate $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz\, dz\, dy\, dx$ [Ans - $\frac{13}{9} - \frac{1}{6}\log 3$]

Ex-10: $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ [Ans - $\frac{4}{35}$], (AKTU-2007).

Ex-11: Evaluate $\int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)\, dz\, dy\, dx$ [Ans - $\frac{\pi^2}{8}$].
(AKTU-2011).

Ex-12: Evaluate $\iiint_R dx\, dy\, dz$, R is a region bounded by the Curve $x+y+z=a$, $x \geq 0, y \geq 0, z \geq 0$. (AKTU-2014).

Soln: Given $x+y+z=a$.

- Put $y=0, z=0$ we get $x=a$
- Put $z=0$, $y=a-x$
- $z=a-x-y$.

Hence $$\iiint_R dx\, dy\, dz = \int_{x=0}^a \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dz\, dy\, dx.$$

$$= \int_{x=0}^a \int_{y=0}^{a-x} [z]_0^{a-x-y}\, dy\, dx$$

$$= \int_{x=0}^a \int_{y=0}^{a-x} [(a-x) - y]\, dy\, dx.$$

$$= \int_{x=0}^a \left[(a-x)y - \frac{y^2}{2} \right]_0^{a-x} dx.$$

$$= \int_{x=0}^a \left[(a-x)^2 - \frac{(a-x)^2}{2} \right] dx$$

$$= \frac{1}{2}\int_{x=0}^a (a-x)^2 dx = \frac{1}{2}\int_{x=0}^a (x-a)^2 dx.$$

$$= \frac{1}{2}\left[\frac{(x-a)^3}{3}\right]_0^a = \frac{1}{2}\left[0 - (-a^3)\right] = \frac{a^3}{6}.$$

Ex-13: If the volume of an object expressed in spherical coordinates as follows:

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2 \sin\phi \, dr\, d\phi\, d\theta .$$ Evaluate V.

(AKTU-2017).

Sol^n: Given $$V = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi \left[\int_{r=0}^{1} r^2\, dr\right] d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi \left[\frac{r^3}{3}\right]_0^1 d\phi\, d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi \, d\phi\, d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{2\pi}\left[-\cos\phi\right]_0^{\pi} d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{2\pi}\left[-\cos\pi + \cos 0\right] d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{2\pi}\left[-(-1)+1\right] d\theta$$

$$= \frac{2}{3}\int_{\theta=0}^{2\pi} d\theta = \frac{2}{3}\left[\theta\right]_0^{2\pi} = \frac{4\pi}{3}.$$

Ex-14: Evaluate

$$\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta .$$ [Ans $\frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$].

Ex-15: Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$ (AKTU-2003, 2005)

[Ans $=\frac{5}{8}$]

Ex-16: Compute $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ if the region of integration is bdd by the coordinate planes & plane $x+y+z=1$ (AKTU-2006, 2007)

[Ans $\frac{1}{2}\log 2 - \frac{5}{16}$]

Change of order of integration→

Some double integration problems seem to be complicated, but can be made easy by changing the order of integration.

Working Rule→

① Write the given limits of integration.

② Draw the region or figure of given limits of integration.

③ In this region, check the strip for variable limits is horizontal or vertical.

④ Draw the same region or figure again.

⑤ ⓐ If the strip in first figure is horizontal the convert it into vertical strip in second figure.

ⓑ If the strip in first figure is vertical then convert it into horizontal strip in second figure.

⑥ Now change the given double integration according to second figure.

Note→ (i) If the limits of x are constant in the given integration the after changing the order of integration limits of y will be constant. (or vice-versa).

(ii) $dx\,dy$ will be chang by $dy\,dx$ or vice-versa.

Ex-1→ Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx$. (AKTU-2011, 2014).

Solⁿ→ Let
$$I=\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx$$

Given limits of integration are
$x=0$ to $x=1$.
& $y=0$ to $y=\sqrt{1-x^2}$ i.e $x^2+y^2=1$

Draw these limits, we get the region of integration is OABO.

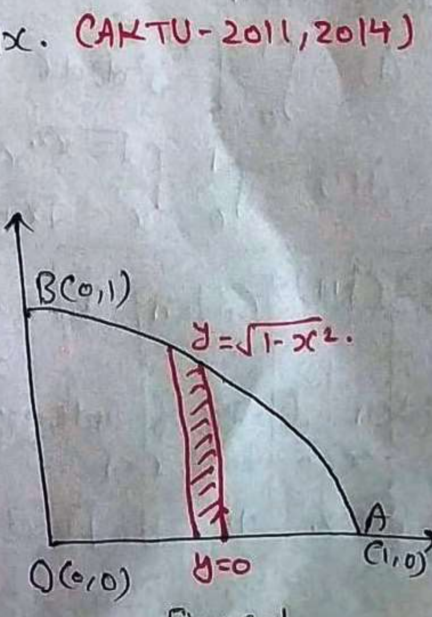


Figure-1

Now by changing the order of integration, we get

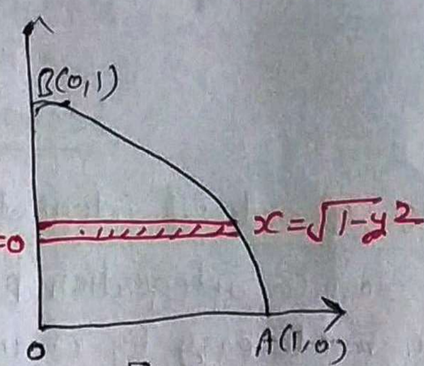


Figure-2

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dx\,dy.$$

$$= \int_{y=0}^{1} \frac{e^y}{e^y+1}\left[\int_{x=0}^{\sqrt{1-y^2}} \frac{dx}{\sqrt{1-y^2-x^2}}\right]dy$$

$$= \int_{y=0}^{1} \frac{e^y}{e^y+1}\left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}} dy \qquad \left[\because \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \int_{y=0}^{1} \frac{e^y}{e^y+1}\cdot\left[\sin^{-1}1 - \sin^{-1}0\right]dy.$$

$$= \frac{\pi}{2}\int_{y=0}^{1} \frac{e^y}{e^y+1}\,dy \qquad \text{Let } e^y+1 = t,\ e^y dy = dt$$

$$= \frac{\pi}{2}\left[\log(e^y+1)\right]_0^1 = \frac{\pi}{2}\left[\log(e+1) - \log 2\right]$$

$$= \frac{\pi}{2}\log\frac{e+1}{2}.$$

Ex-2 Evaluate the following integrals by changing the order of integration

(i) $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$ (Ans $\frac{\pi}{6}$) (ii) $\int_0^4\int_y^4 \frac{x\,dx\,dy}{x^2+y^2}$ [Ans π]

(iii) $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ (AKTU-2012). $1-\frac{1}{\sqrt{2}}$ (iv) $\int_0^a\int_0^{\sqrt{a^2-x^2}} dy\,dx$.

Ex-3: Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and hence evaluate the same. (AKTU- 2010, 2014, 2015, 2016, 2017, 2019)

Solⁿ: Given

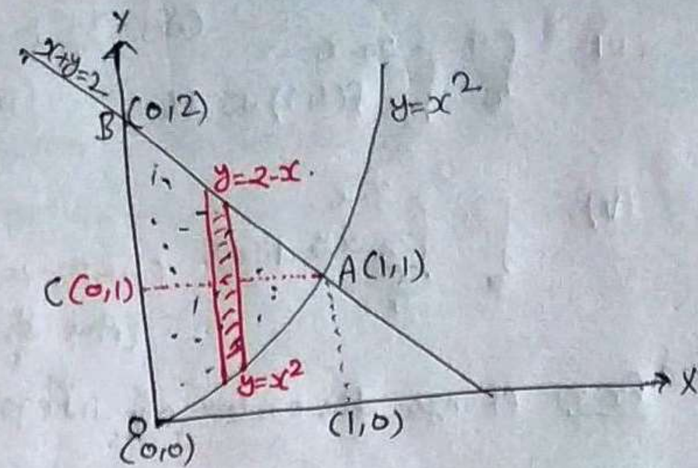
$$I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy\, dy\, dx$$

The given limits of integration are

$x = 0$ to $x = 1$

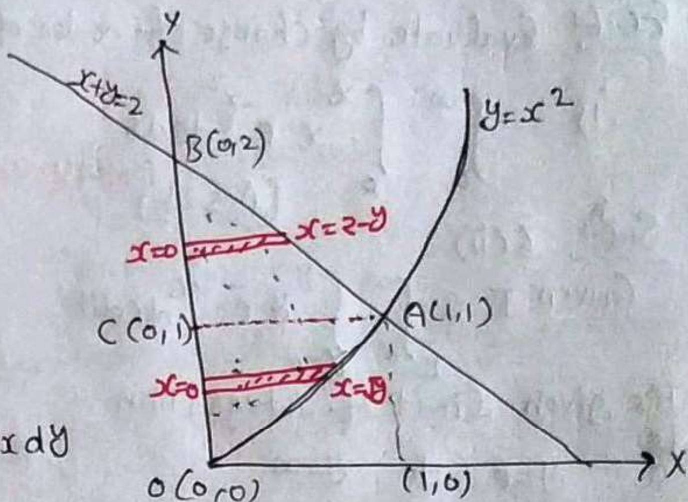
& $y = x^2$ to $y = 2-x$

i.e. $x + y = 2$.



Draw the limits, we get the region of integration OABCO.

Now by changing the order of integration, we get



Region OABCO = OACO + CABC

$$I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy\, dy\, dx$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y \left[\int_{x=0}^{\sqrt{y}} x\, dx \right] dy + \int_{y=1}^{2} y \left[\int_{x=0}^{2-y} x\, dx \right] dy$$

$$= \int_{y=0}^{1} y \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy + \int_{y=1}^{2} y \left[\frac{x^2}{2} \right]_0^{2-y} dy$$

$$= \frac{1}{2} \int_{y=0}^{1} y^2\, dy + \frac{1}{2} \int_{y=1}^{2} y(2-y)^2\, dy = \frac{1}{2} \int_{y=0}^{1} y^2\, dy + \frac{1}{2} \int_{y=1}^{2} (4y + y^3 - 4y^2)\, dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1 + \frac{1}{2} \left[4\frac{y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3} \right]_{y=1}^{2}$$

$$= \frac{1}{2}\left(\frac{1}{3} - 0\right) + \frac{1}{2} \left[2 \times 4 + \frac{16}{4} - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3} \right]$$

$$= \frac{1}{6} + \frac{1}{2}\left(10 - \frac{32}{3} - \frac{1}{4} + \frac{4}{3}\right) = \frac{3}{8}.$$

*Ex-4: Evaluate by changing the order of integration

(i) $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ (AKTU-2010) (Ans - $\frac{3a^4}{8}$)

(ii) $\int_0^1 \int_{y^2}^{2-y} xy\,dx\,dy$. (AKTU-2014). (Ans - $\frac{3}{8}$).

(iii) $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\,dy\,dx$

(iv) $\int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-x} (x^2+y^2)\,dy\,dx$. (AKTU-2011) [Ans $\frac{314}{35}a^4$].

(v) $\int_0^2 \int_{\frac{x^2}{4}}^{3-x} xy\,dy\,dx$ (AKTU-2018) [Ans $\frac{8}{3}$].

Ex-5: Change the order of integration $\int_0^1 \int_{x^2}^{2-x} f(x,y)\,dy\,dx$. (AKTU-2018).

Ex-6: Evaluate by change of order of integration

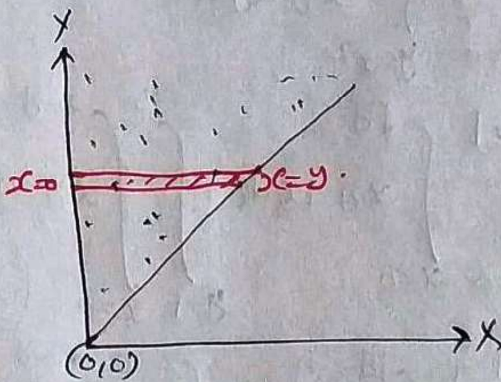
(i) $\int_0^\infty \int_0^x x\,e^{-\frac{x^2}{y}}\,dy\,dx$ (AKTU-2013) (Ans $=\frac{1}{2}$)

(ii) $\int_0^\infty \int_0^y y\,e^{-y^2/x}\,dx\,dy$. (AKTU-2014).

Solⁿ: 6(ii)

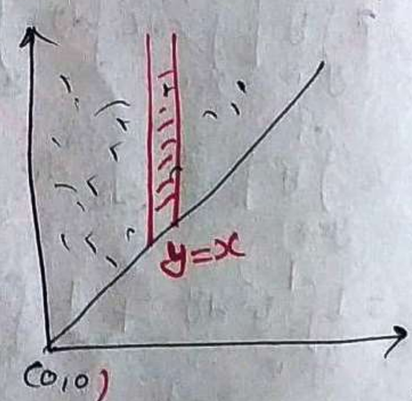
Given $I = \int_{y=0}^\infty \int_{x=0}^y y\,e^{-y^2/x}\,dx\,dy$.

The given limit of integration is
$y=0$ to $y=\infty$
& $x=0$ to $x=y$



Draw these limits, we get the region of integration.

By changing the order of integration we get



$$I = \int_{x=0}^\infty \int_{y=x}^\infty \cancel{y}\, e^{-y^2/x}\,dy\,dx.$$

[Let $+\frac{y^2}{x}=t$ then $+2y\,dy = x\,dt$]

when $y=x$ then $t=+x$
$y=\infty$, then $t=\infty$

$$= \int_{x=0}^\infty \left[\frac{x}{2}\int_{t=+x}^\infty e^{-t}\,dt\right]dx$$

$$= \int_{x=0}^\infty \frac{x}{2}\left[\frac{e^{-t}}{(-1)}\right]_x^\infty dx.$$

$$= \frac{1}{2}\int_0^\infty x\,e^{-x}\,dx = \frac{1}{2}\left[x\frac{e^{-x}}{(-1)} - e^{-x}\right]_0^\infty = \frac{1}{2}\left[0-0+e^0\right] = \frac{1}{2}.$$

Ex-7→ Evaluate by changing the order of integration
$$\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dy\,dx. \quad (\text{AKTU-2008}). \quad (\text{Ans}-1).$$

Ex-8→ Evaluate $\int_0^3\int_0^{6/x} x^2\,dy\,dx.$ (AKTU-2015). (Ans-27).

Ex-9→ Changing the order of integration in the double integral
$$I=\int_0^8\int_{x/4}^{2} f(x,y)\,dy\,dx \text{ leads to the value } I=\int_r^s\int_p^q f(x,y)\,dx\,dy.$$
What is the value of q?. (AKTU-2017).

Ex-10→ Evaluate by change of its order

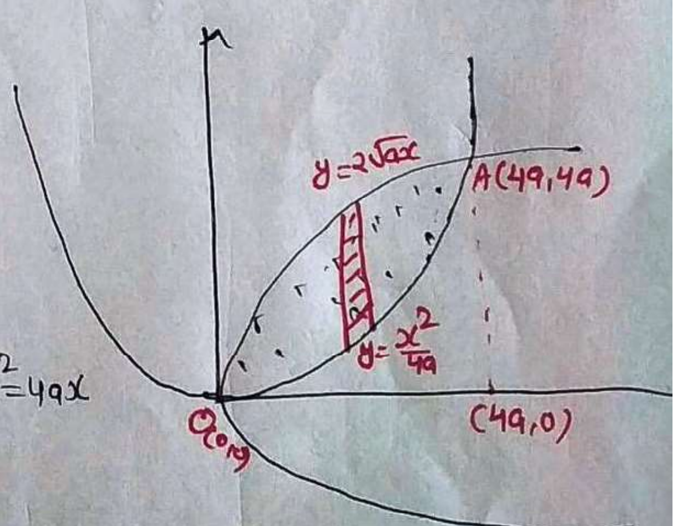
(i) $\int_{y=0}^{1}\int_y^1 x^2\cos(x^2-xy)\,dx\,dy$ (~~AKT~~

(ii) $\int_0^1\int_x^1 \sin y^2\,dy\,dx.$ (AKTU - 2015).
Ans→ $\frac{1}{2}(1-\cos 1)$.

(iii) $\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$. $\left[\text{Ans } \frac{\pi a}{4}\right]$.

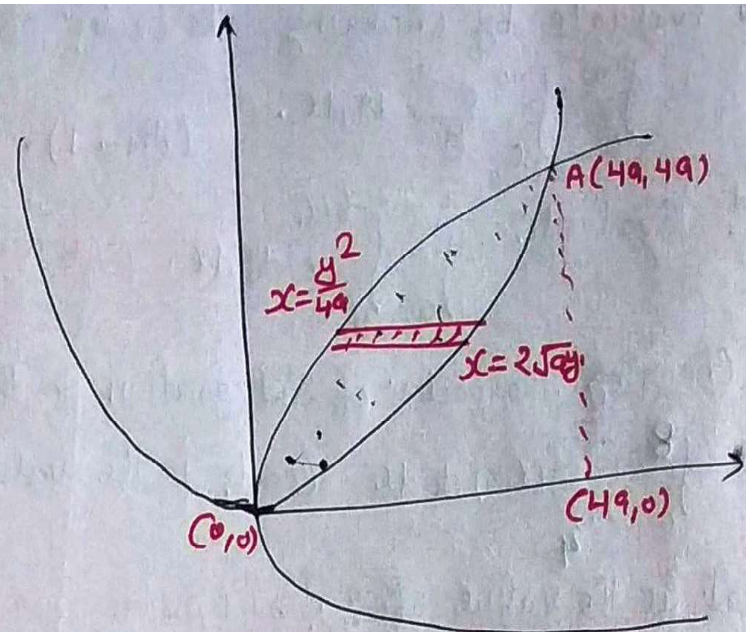
Ex-11→ Change the order of integration in the following integral and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx.$ (AKTU-2008).

Solⁿ→ Given
$$I=\int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$



Given limits of integration are
$x=0$ to $x=4a$
$y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$ i.e $x^2=4ay$ to $y^2=4ax$

Now by changing the order of integration we get,



$$I = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$

$$= \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy$$

$$= \int_{y=0}^{4a} \left[x \right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy = \int_{y=0}^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a} \right] dy.$$

$$= 2\sqrt{a} \left(\frac{y\sqrt{y}}{3/2} \right)_0^{4a} - \frac{1}{4a} \left(\frac{y^3}{3} \right)_0^{4a}.$$

$$= 2\sqrt{a} \times \frac{2}{3} \times 4a \times 2\sqrt{a} - \frac{1}{4a} \times \frac{(4a)^3}{3}$$

$$= \frac{32}{3}a^2 - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

MATHEMATICS-I

KAS-103T

* Lecture No-37 *

Module-IV

[redacted]

Change of variable→

or Double integral by changing cartesian coordinates (x,y) into polar coordinates (r,θ) ⟶

To transform cartesian coordinate (x,y) into polar coordinate (r,θ), we define

$x = r\cos\theta$ where $x^2+y^2=r^2$

$y = r\sin\theta$

& $dx\,dy = J\,dr\,d\theta = \dfrac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= r(\cos^2\theta + \sin^2\theta)\,dr\,d\theta$$

$$= r\,dr\,d\theta$$

$$\Rightarrow \boxed{dx\,dy = r\,dr\,d\theta}$$

Hence $$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Rule→ ① Write given limits of integration & draw region.

② Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ in given integration.

③ Find the limit of r & θ.

④ Solve the final integration.

Ex-1→ Evaluate the integral by changing into polar form

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy \,. \quad \text{(CAKTU-2012)}$$

Solⁿ→ Given $$\int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$$

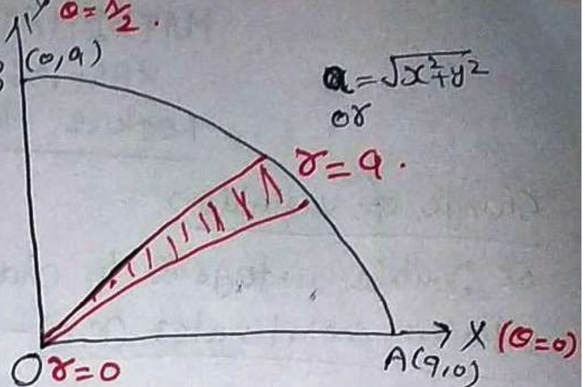
Given limit of integration are

$0 \le y \le a$ & $x=0$ to $x=\sqrt{a^2-y^2}$ i.e $x^2+y^2=a^2$.

Put

$x = r\cos\theta$, $y = \sin\theta$, $x^2+y^2 = r^2$

and $dx\,dy = r\,d\theta\,d\theta$ in the given integration, we get



$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 . r\,dr\,d\theta$$

[$\because$ $x^2+y^2 = a^2$
then $r^2 = a^2$
$\Rightarrow r = a$].

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{a} r^3 dr \right] d\theta = \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4} \right]_0^a d\theta$$

$$= \frac{1}{4}a^4 \int_0^{\pi/2} d\theta = \frac{1}{4}a^4 \left[\theta\right]_0^{\pi/2} = \frac{1}{4}a^4 \times \frac{\pi}{2} = \frac{\pi a^4}{8}.$$

Ex-2) Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing to polar coordinates. (AKTU - 2007, 2008, 2014).

Sol^n ~~Given~~ Let $I = \int_{x=0}^{2} \int_{y=0}^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$

Given limit of integration are

$x=0$ to $x=2$

& $y=0$ to $y = \sqrt{2x-x^2}$

or

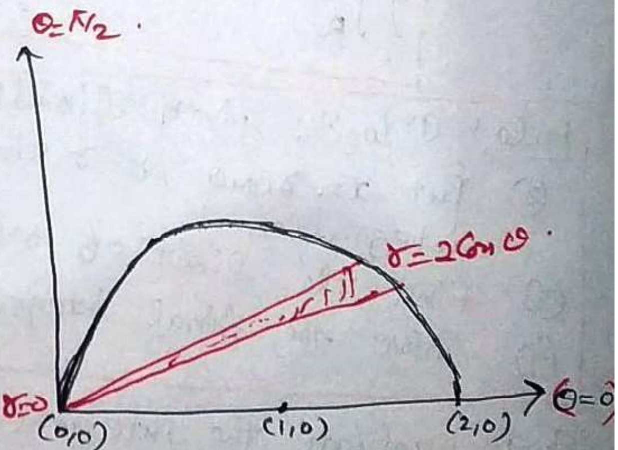
$x^2+y^2 = 2x$

or

$(x^2-2x+1)+y^2 = 1$

or

$(x-1)^2+(y-0)^2 = 1$.



Put $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

and $dx\,dy = r\,dr\,d\theta$

in the given integration, we get.

[we have $x^2+y^2 = 2x$
$\Rightarrow r^2 = 2r\cos\theta$
$\Rightarrow r = 2\cos\theta$]

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{\sqrt{r^2}}\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} r\cos\theta\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^2}{2}\right]_0^{2\cos\theta} \cdot \cos\theta\, d\theta .$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{4\cos^2\theta}{2} - 0\right] \cos\theta\, d\theta .$$

$$= 2\int_{\theta=0}^{\pi/2} \cos^3\theta\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \cos^3\theta \sin^0\theta\, d\theta \qquad \left[\int_{\theta=0}^{\pi/2} \cos^m\theta \sin^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right]$$

$$= 2\,\frac{\Gamma\frac{3+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{3+0+2}{2}}$$

$$= 2\,\frac{\Gamma 2\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{2}} \qquad \left[\because \Gamma n = \lfloor n-1 \quad \Gamma n+1 = n\Gamma n \quad \& \ \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

$$= \frac{\lfloor 1 \ \sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}$$

$$= \frac{\sqrt{\pi}}{\frac{3}{4}\sqrt{\pi}} = \frac{4}{3} .$$

Ex-3) Change into polar coordinates and evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx$. Hence show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

(AKTU-2015).

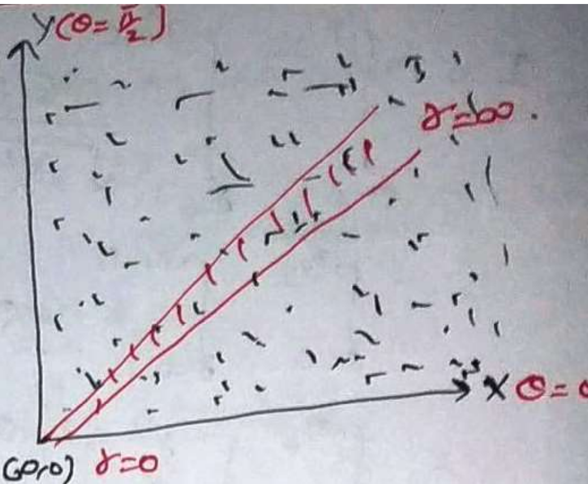
Solⁿ) Let $I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dy\, dx .$

Given limit of integration are

$x=0$ to $x=\infty$

$y=0$ to $y=\infty$

Draw thes limits.



Put $x=r\cos\theta$, $y=r\sin\theta$

$x^2+y^2=r^2$

& $dx\,dy = r\,dr\,d\theta$ in the given integral we get,

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r \cdot dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{\infty} e^{-r^2} r\,dr\right] d\theta$$

Let $r^2=t$

then $2r\,dr = dt$

$r\,dr = \frac{dt}{2}$.

& when $r=0$, $t=0$

$r=\infty$, $t=\infty$]

$$= \int_{\theta=0}^{\pi/2}\left[\frac{1}{2}\int_{t=0}^{\infty} e^{-t}\,dt\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-t}}{(-1)}\right]_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-\infty}}{(-1)} - \frac{e^{0}}{(-1)}\right] d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2}(0+1)\,d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{\pi}{4} \longrightarrow (*)$$

Now, let $I = \int_0^{\infty} e^{-x^2}\,dx \longrightarrow (1)$

Then, we get $I = \int_0^{\infty} e^{-y^2}\,dy \longrightarrow (2)$ $\left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$

From (1) & (2),

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-x^2} \times e^{-y^2}\,dx\,dy = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \frac{\pi}{4}. \quad \text{(from *)}$$

$$\Rightarrow I^2 = \frac{\pi}{4}$$

$$\Rightarrow \boxed{I = \frac{\sqrt{\pi}}{2} = \int_0^{\infty} e^{-x^2}\,dx}$$

Ex-4) Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$ by changing into polar coordinates. (AKTU-2014) [Ans $\frac{3\pi a^4}{4}$].

Ex-5) Evaluate the following by changing into polar-coordinates:-

i) $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$ (Ans πa^2).

[Hint $0 \le \theta \le 2\pi$, $0 \le r \le a$].

(ii) $\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ [Ans $\frac{\pi a}{4}$].

Ex-6) Evaluate $\iint \sin[\pi(x^2+y^2)]\,dx\,dy$ over the region bdd by the circle $x^2+y^2=1$ by changing to polar coordinates. [Ans - 2].

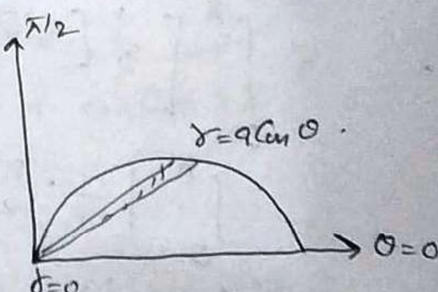
Ex-7) Evaluate $\iint (x^2+y^2)^{7/2}\,dx\,dy$ over the circle $x^2+y^2=1$.

[Ans $\frac{2\pi}{9}$] (AKTU-2008)

Ex-8) Evaluate the following by changing into polar coordinates: $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$. (AKTU-2009)

[Ans $\frac{\pi a^5}{20}$].

Ex-9) Evaluate $\iint \sqrt{a^2-x^2-y^2}\,dx\,dy$ over the semicircle $x^2+y^2=ax$ in the +ve quadrant. [Hint



[Hint $\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\,.\,r\,dr\,d\theta$]

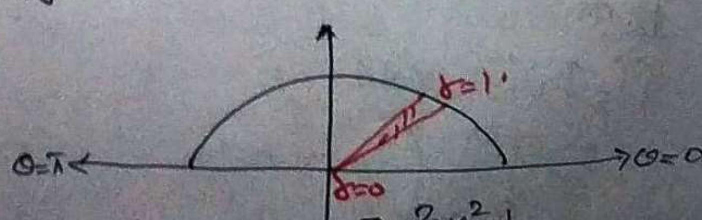
[Ans - $\frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$.

Ex-10) Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the +ve quadrant of the circle $x^2+y^2=1$. [Ans - $\frac{\pi^2}{8}-\frac{\pi}{4}$] (AKTU-2005).

Ex-11) Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ over the semi-circle $x^2+y^2=1$.

Soln)

Let $I = \iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}$



[$x^2+y^2=1$
$\Rightarrow r^2=1 \Rightarrow r=1$].

Put $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$ & $dx\,dy = r\,dr\,d\theta$
in given integral, we get

$$I = \int_{\theta=0}^{\pi}\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}} \cdot r\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{1} \frac{(1-r^2)}{\sqrt{1+r^2}\sqrt{1-r^2}}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{1} \frac{(r-r^3)}{\sqrt{1-r^4}}\, dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\int_{r=0}^{1} \frac{r}{\sqrt{1-r^4}}\,dr - \int_{r=0}^{1}\frac{r^3}{\sqrt{1-r^4}}\,dr\right]d\theta$$

Let $r^2 = t$
then $r\,dr = \frac{dt}{2}$
& $r=0,\ t=0$
$r=1,\ t=1$

Let $1-r^4 = u$
$-4r^3\,dr = du$
$\Rightarrow r^3\,dr = \frac{-du}{4}$
when $r=0$, $u=1$
$r=1$, $u=0$.

$$\Rightarrow I = \int_{\theta=0}^{\pi}\left[\frac{1}{2}\int_{t=0}^{1}\frac{dt}{\sqrt{1-t^2}} - \left(\frac{-1}{4}\right)\int_{u=1}^{0}\frac{1}{\sqrt{u}}\,du\right]d\theta$$

$$= \int_{\theta=0}^{\pi}\left[\frac{1}{2}\left[\sin^{-1}t\right]_0^1 + \frac{1}{4}\left[\frac{\sqrt{u}}{1/2}\right]_1^0\right]d\theta$$

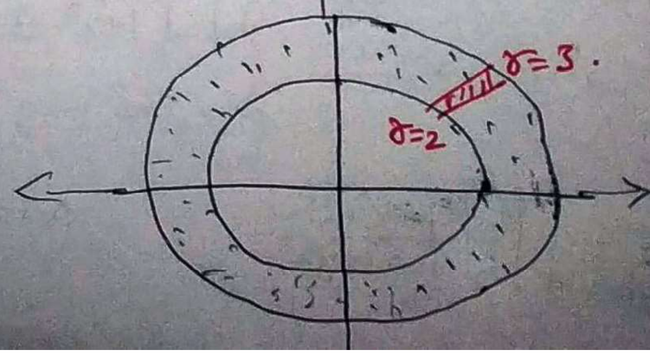
$$= \int_{\theta=0}^{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{4}\times 2(-1)\right]d\theta$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right)\int_{\theta=0}^{\pi} d\theta = \left(\frac{\pi}{4}-\frac{1}{2}\right)\pi = \frac{\pi^2}{4} - \frac{\pi}{2}.$$

Ex-11 Evaluate the integral $\iint_R \sqrt{x^2+y^2}\,dx\,dy$ by changing to polar coordinates where R is the region in the xy-plane bdd by the circles $x^2+y^2=4$ & $x^2+y^2=9$.

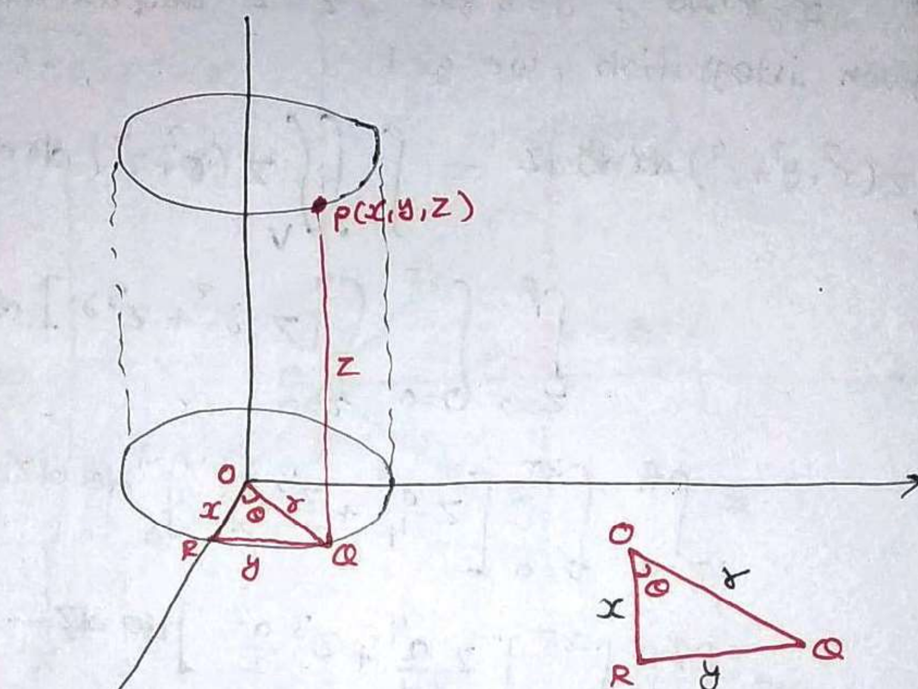
Hints

$$\iint_R \sqrt{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=2}^{3} r\cdot r\,dr\,d\theta = \frac{38\pi}{3}.$$



⊛ Change of Cartesian coordinates into cylinderical coordinates →

Change cartesian coordinates (x, y, z) into cylinderical coordinates (r, θ, z).



$$\cos\theta = \frac{B}{H} = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{P}{H} = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$z = z.$$

or

$$r = \sqrt{x^2+y^2}.$$

$$\theta = \tan^{-1} y/x$$

$$z = z.$$

where z is the height of the cylinder and

r is the radius of circle & $0 \le \theta \le 2\pi$.

Also
$$dx\,dy\,dz = J\,dr\,d\theta\,dz$$
$$= r\,dr\,d\theta\,dz$$

$$\left[\because J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r. \right]$$

Hence

$$\boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.}$$

Ex-1) Evaluate $\iiint_V z(x^2+y^2+z^2)\,dx\,dy\,dz$, over the volume of the cylinder $x^2+y^2=a^2$ intercepted by the plane $z=0$ & $z=h$.

Sol^n) Put $x=r\cos\theta$, $y=r\sin\theta$, $z=z$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz$ in the given integration, we get

$$\iiint_V z(x^2+y^2+z^2)\,dx\,dy\,dz = \iiint_V z(r^2+z^2)\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} [z\,r^3 + z^3 r]\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\left[z\frac{r^4}{4} + z^3\frac{r^2}{2}\right]_0^a d\theta\,dz$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\left[z\frac{a^4}{4} + z^3\frac{a^2}{2}\right]d\theta\,dz.$$

$$= \int_{z=0}^{h}\left[z\frac{a^4}{4} + z^3\frac{a^2}{2}\right]\left[\int_{\theta=0}^{2\pi} d\theta\right]dz$$

$$= \int_{z=0}^{h}\left[z\frac{a^4}{4} + z^3\frac{a^2}{2}\right]\cdot 2\pi\,dz.$$

$$= 2\pi\frac{a^2}{2}\int_{z=0}^{h}\left[z\frac{a^2}{2} + z^3\right]dz.$$

$$= a^2\pi\left[\frac{z^2}{2}\cdot\frac{a^2}{2} + \frac{z^4}{4}\right]_0^h$$

$$= a^2\pi\left[\frac{a^2h^2}{4} + \frac{h^4}{4}\right] = \frac{a^2\pi}{4}h^2(a^2+h^2).$$

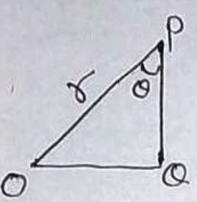
Ex-2) Evaluate $\iiint z(x^2+y^2)\,dx\,dy\,dz$ over the volume of the cylinder $x^2+y^2=1$ intercepted by the planes $z=2$ & $z=3$.

[Ans $\frac{5\pi}{4}$]

$$\boxed{\text{Hint)}\ \int_{z=2}^{3}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}}$$

Change of Cartesian coordinates (x, y, z) into spherical polar coordinates $(r, \theta, \phi) \rightarrow$

Here $OP = \sqrt{x^2+y^2+z^2} = r.$

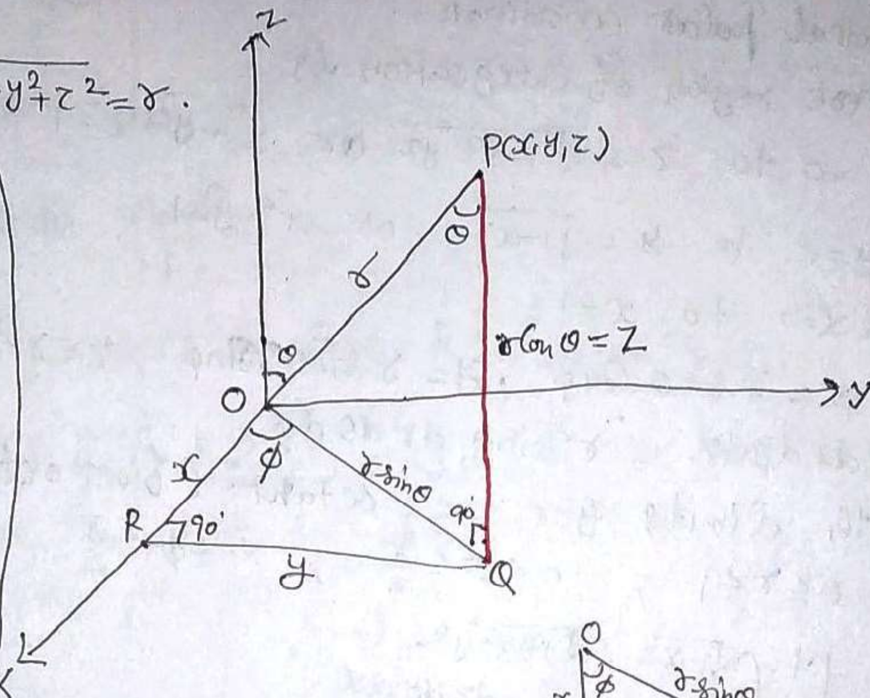


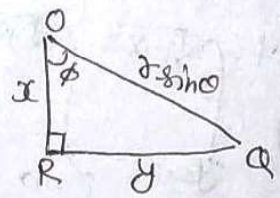
$\cos\theta = \frac{B}{H} = \frac{PQ}{r}$

$\Rightarrow PQ = r\cos\theta$

$\sin\theta = \frac{P}{H} = \frac{OQ}{r}$

$\Rightarrow OQ = r\sin\theta$





Then $\sin\phi = \frac{P}{H} = \frac{y}{r\sin\theta}$

$\cos\phi = \frac{B}{H} = \frac{x}{r\sin\theta}$

$\therefore \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \ \& \ z = r\cos\theta.$

$$\& \ J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = r^2\sin\theta \quad \text{(try yourself)}$$

$\Rightarrow dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

Hence
$$\boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi.}$$

Note →

For complete sphere	for hemisphere	For +ve octant of sphere
$r=0$ to $r=a$	$r=0$ to $r=a$	$r=0$ to $r=a$
$\theta=0$ to $\theta=\pi$	$\theta=0$ to $\theta=\pi/2$	$\theta=0$ to $\theta=\pi/2$
$\phi=0$ to $\phi=2\pi$	$\phi=0$ to $\phi=2\pi$	$\phi=0$ to $\phi=\frac{\pi}{2}$

Ex-11 Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical polar coordinates.

Sol^n: Here region of integration is

$z=0$ to $z=\sqrt{1-x^2-y^2}$ or $x^2+y^2+z^2=1$

$y=0$ to $y=\sqrt{1-x^2}$ or $x^2+y^2=1$

& $x=0$ to $x=1$.

Put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

& $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

with limits for +ve octant (first octant)

$0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$.

Hence $$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}}$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{1}\left[\frac{1}{\sqrt{1-r^2}}-\sqrt{1-r^2}\right]dr\right]\sin\theta\,d\theta\,d\phi$$

$$\left[\because \frac{r^2}{\sqrt{1-r^2}} = \frac{1}{\sqrt{1-r^2}}-\sqrt{1-r^2}\right]$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\sin^{-1}r - \frac{1}{2}r\sqrt{1-r^2} - \frac{1}{2}\sin^{-1}r\right]_0^1 \sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\sin\theta\,d\theta\,d\phi$$

$$= \frac{\pi}{4}\int_{\phi=0}^{\pi/2}\left[\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\right]d\phi = \frac{\pi}{4}\int_{\phi=0}^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2}d\phi.$$

$$= \frac{\pi}{4}\int_{\phi=0}^{\pi/2}d\phi = \frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Ex-2: Evaluate $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)}$ taken thought the volume of the sphere $x^2+y^2+z^2=a^2$. (AKTU - 2001, 2002)

[Hint $I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \frac{1}{r^2}\cdot r^2 \sin\theta \, dr\,d\theta\,d\phi = 4\pi a$].

Ex-3: Evaluate $\iiint \sqrt{x^2+y^2}$ ~~o~~

$\iiint_R dx\,dy\,dz$ where $R: x^2+y^2+z^2=9$.

Ex-4: Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ ove the first octant of the sphere $x^2+y^2+z^2=a^2$. (ATU - 2007).

Hint :- $\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 \cdot r^2 \sin\theta \, dr\,d\theta\,d\phi$

$= \pi \frac{a^5}{10}$.

Ex-5: Evaluate the integral

$\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$.

[Hint - $\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2 (r^2 \sin\theta \, dr\,d\theta\,d\phi) = \frac{4\pi}{5}$.

Ex-6: Evaluate $\iiint \frac{z^2\,dx\,dy\,dz}{x^2+y^2+z^2}$ over the volume of the sphere $x^2+y^2+z^2=2$.

(AKTU-2004, 2005).

Hint: $I = 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{\sqrt{2}}$

[I = 8 × Integration in first octant].

Ex-7: Evaluate $\iiint xyz\,(x^2+y^2+z^2)\,dx\,dy\,dz$ over the first octant of the sphere $x^2+y^2+z^2=a^2$. (AKTU - 2009, 2005, 2004, 2002)

Soln: Let $I = \iiint xyz\,(x^2+y^2+z^2)\,dx\,dy\,dz$.

Put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

& $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$. we get

[for first octant]

$$I = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)\, r^2 (r\cos\theta)(r^2\sin\theta)\,dr\,d\theta\,d\phi .$$

$[\because x^2+y^2+z^2 = r^2 .$

$$= \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} (\sin^3\theta\cos\theta)(\sin\phi\cos\phi)\left[\int_{r=0}^{a} r^7\,dr\right] d\theta\,d\phi$$

$$\cancel{= \int_{\phi=0}^{\pi/2}(\sin} = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} (\sin^3\theta\cos\theta)(\sin\phi\cos\phi)\left[\frac{r^8}{8}\right]_0^a d\theta\,d\phi$$

$$= \frac{a^8}{8}\int_{\phi=0}^{\pi/2} \sin\phi\cos\phi \left[\int_{\theta=0}^{\pi/2}\sin^3\theta\cos\theta\,d\theta\right] d\phi .$$

$$= \frac{a^8}{8}\int_{\phi=0}^{\pi/2} \sin\phi\cos\phi \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} d\phi .$$

$$= \frac{a^8}{32}\int_{\phi=0}^{\pi/2} \sin\phi\cos\phi\,d\phi$$

$$= \frac{a^8}{32}\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}$$

$$= \frac{a^8}{32}\times\frac{1}{2} = \frac{a^8}{64} .$$

Application of double integration →
or Area by double integration →

Cartesian form →

(*) The area A of a region bdd by the curves $y = f_1(x)$ and $y = f_2(x)$ & the lines $x = a$, $x = b$ is given by

$$\boxed{A = \int_{a}^{b} \int_{y=f_1(x)}^{f_2(x)} dy\, dx.}$$

(*) The area A of a region bdd by the curves $x = f_1(y)$, $x = f_2(y)$ and line $y = c$, $y = d$ is given by

$$\boxed{A = \int_{y=c}^{d} \int_{x=f_1(y)}^{f_2(y)} dx\, dy.}$$

Ex-1: find the area lying b/w the parabola $y = 4x - x^2$ and the line $y = x$. (AKTU-2019).

Sol^n Given curves are

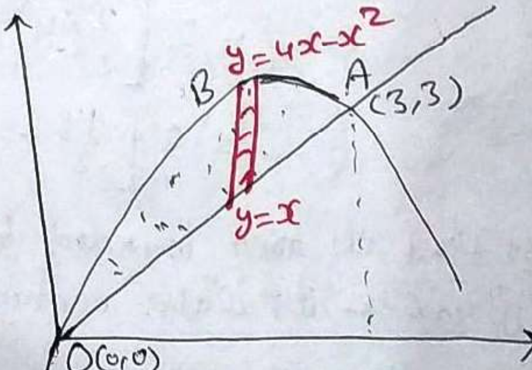
$y = 4x - x^2$ & $y = x$.

On solving,

$$x = 4x - x^2 \Rightarrow x^2 - 3x = 0$$
$$\Rightarrow x(x-3) = 0$$
$$\Rightarrow x = 0, x = 3.$$

Then $y = 0$, $y = 3$.

∴ Intersecting points are $(0,0)$ & $(3,3)$.



∴ Required Area $= \iint_{OABO} dy\, dx$

$$= \int_{x=0}^{3} \int_{y=x}^{4x-x^2} dy\, dx.$$

$$= \int_{x=0}^{3} [y]_{x}^{4x-x^2} dx = \int_{x=0}^{3} [4x - x^2 - x]\, dx$$

$$= \int_{x=0}^{3} (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = 4.5.$$

[Note $y = 4x - x^2$
$\Rightarrow y = -(x^2 - 4x)$
$\Rightarrow y - 4 = -(x^2 - 4x + 4)$
$\Rightarrow y - 4 = -(x-2)^2$
$\Rightarrow Y = -X^2$
i.e a parabola $X^2 = -Y$ with the vertex $(2,4)$]

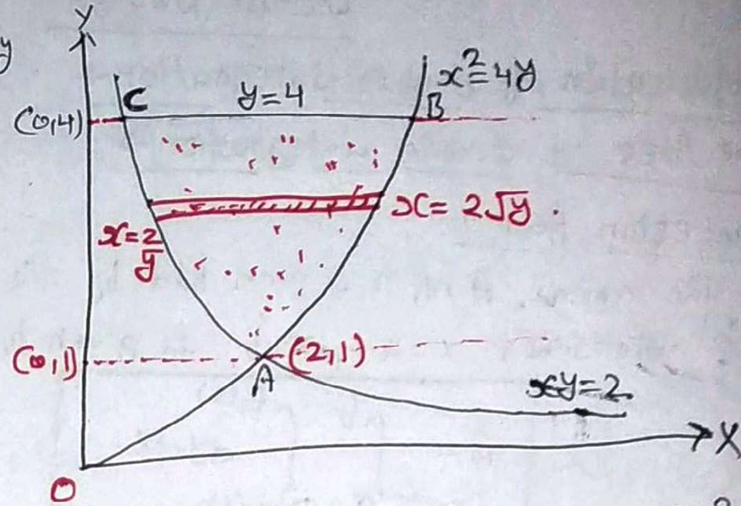
Ex-2: Determine the area of region bdd by the curves $xy=2$, $4y=x^2$, $y=4$. (AKTU-2014, 2015, 2016).

Sol^n: Given region is bdd by the curves

$xy=2$

$x^2=4y$

$y=4$



Hence required

$$\text{Area} = \iint_{ABCA} dx\,dy$$

$$= \int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\,dy$$

[on solving $xy=2$ & $x^2=4y$ we get $x^2=4\cdot\frac{2}{x}$ $\Rightarrow x^3=8 \Rightarrow \boxed{x=2}$ & $\boxed{y=1}$]

$$= \int_{y=1}^{4} [x]_{2/y}^{2\sqrt{y}}\,dy = \int_{y=1}^{4} \left[2\sqrt{y} - \frac{2}{y}\right] dy$$

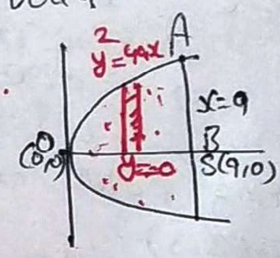
$$= \left[2\frac{y^{3/2}}{3/2} - 2\log y\right]_1^4 = 2\left[\frac{2}{3}y^{3/2} - \log y\right]_1^4.$$

$$= 2\left[\frac{2}{3}(4^{3/2} - 1^{3/2}) - (\log 4 - \log 1)\right]$$

$$= 2\left[\frac{14}{3} - \log 4\right] = \frac{28}{3} - 4\log 2. \quad \underline{\text{Ans}}.$$

Ex-3: Find the area bounded by the parabola $y^2=4ax$ & its latus rectum. (AKTU-2012).

Sol^n: Given $y^2=4ax$



Required Area $= 2\times(\text{Area of } OABO)$

$$= 2\iint_{OABO} dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} dy\,dx$$

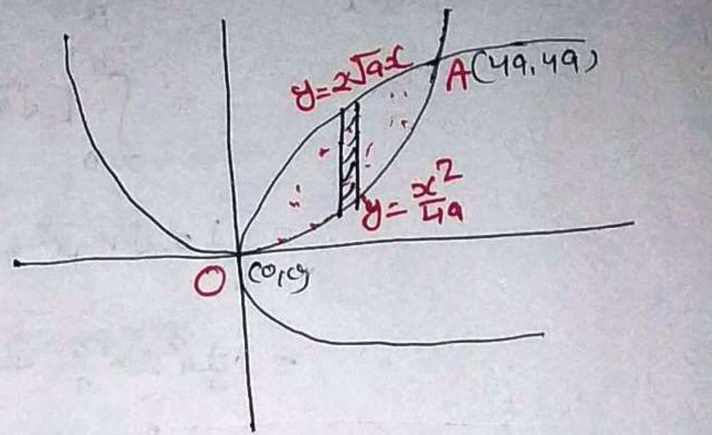
$$= 2\int_{x=0}^{a} 2\sqrt{ax}\,dx = 4\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_0^a = \frac{8a^2}{3}.$$

Ex-4: ~~find~~ Show, by double integration, that the area b/w the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$. (AKTU-2015).

Solⁿ: Given

$$y^2=4ax \rightarrow (1)$$

$$\& \quad x^2=4ay \rightarrow (2)$$



from (2), $y=\frac{x^2}{4a} \rightarrow (3)$

put in (1), we get

$$\frac{x^4}{16a^2}=4ax$$

$$\Rightarrow x^4-64a^3x=0$$

$$\Rightarrow x(x^3-64a^3)=0$$

$$\Rightarrow x=0,\ x=4a.$$

Then from (3), $y=0$, $y=4a$.

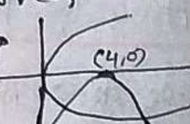
Therefore the point of intersection are O(0,0) & A(4a, 4a)

Hence Area b/w the given parabolas

$$A=\iint dy\,dx=\int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx=\frac{16a^2}{3}.$$

(Prove yourself)

Ex-5: find the area of the region occupied by the curves $y^2=x$ & $y^2=4-x$. (AKTU-2013). $\left(\frac{16\sqrt{2}}{3}\right)$ →



Ex-6: find the area of the circle. $x^2+y^2=a^2$.

Ex-7: find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (Ans – πab).

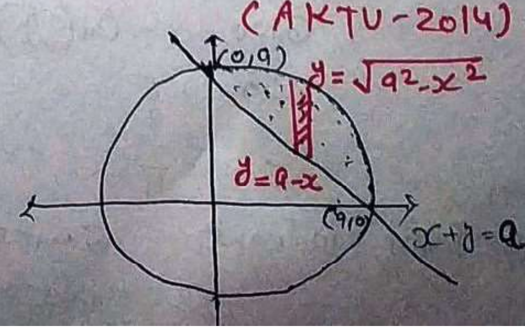
Ex-8: (i) find, by double integration, the smaller areas bdd by the circle $x^2+y^2=9$ and the line $x+y=3$. [Ans $\frac{9}{2}(\pi-2)$.

(ii) Find, by double integration, the area of the region enclosed by the curves $x^2+y^2=a^2$, $x+y=a$ in the first quadrant. (AKTU-2014).

Solⁿ: 8(ii) Given curve are

$$x^2+y^2=a^2, \quad x+y=a.$$

Required Area $=\iint dy\,dx$



$$= \int_{x=0}^{a} \int_{y=a-x}^{\sqrt{a^2-x^2}} dy \, dx = \int_{x=0}^{a} \left[y \right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \left[\sqrt{a^2-x^2} - (a-x) \right] dx.$$

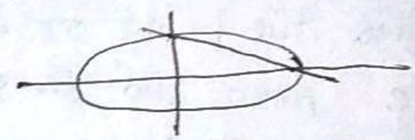
$$= \left[\frac{1}{2} x \sqrt{a^2-x^2} + \frac{1}{2} a^2 \sin^{-1}\frac{x}{a} \right]_0^a - \left[\frac{(a-x)^2}{-2} \right]_0^a$$

$$= \frac{1}{2} a^2 \sin^{-1}\frac{a}{a} + \frac{1}{2}\left[0 - a^2 \right]$$

$$= \frac{1}{2} a^2 \cdot \sin^{-1} 1 + \frac{1}{2} \times -a^2$$

$$= \frac{1}{2} a^2 \left(\frac{\pi}{2} - 1 \right).$$

Ex-9: Find the smaller of the areas bdd by the ellipse $4x^2 + 9y^2 = 36$ & the straight line $2x + 3y = 6$.



[Ans - $\frac{3}{2}(\pi - 2)$.

Ex-10: Find area enclosed b/w the parabola $y = x^2$ & the straight line $y = x$. (AKTU-2018). [Ans $\frac{1}{6}$]

Area by Polar curve

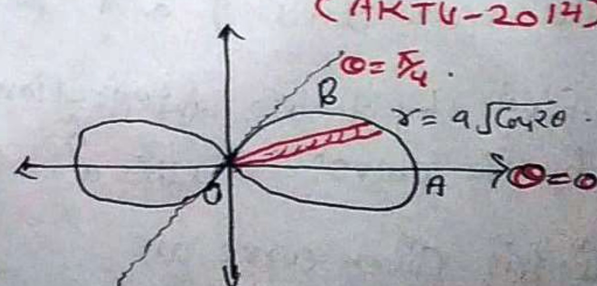
The area A of the region bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the line $\theta = \alpha$, $\theta = \beta$ is

$$\boxed{A = \int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} r \, dr \, d\theta}$$

Ex-11: Compute the area bounded by the lemniscate $r^2 = a^2 \cos 2\theta$. (AKTU-2014).

Sol^n: Given

$r^2 = a^2 \cos 2\theta$



Put $r = 0$, $a^2 \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0 = \cos\left(\pm\frac{\pi}{2}\right)$

$\Rightarrow 2\theta = \pm\frac{\pi}{2}$

$\Rightarrow \theta = \pm\frac{\pi}{4}$.

For first quadrant θ lies from 0 to $\frac{\pi}{4}$

& r lies from 0 to $a\sqrt{\cos 2\theta}$.

Hence required Area $= 4 \times$ Area in first quadrant

$$= 4 \times \iint_{OABO} r\,dr\,d\theta .$$

$$= 4 \times \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta .$$

$$= 4 \int_{\theta=0}^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2 \int_{\theta=0}^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

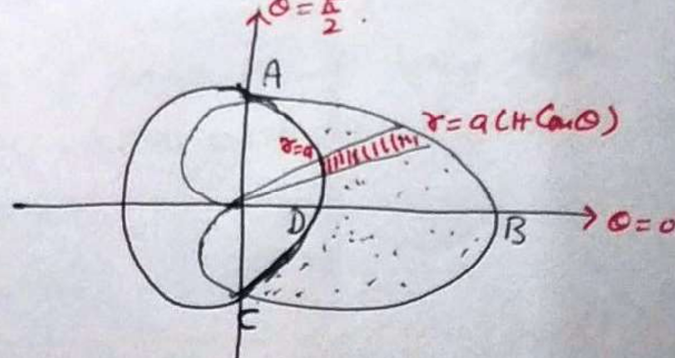
$$= a^2 \left(\sin\frac{\pi}{2} - 0\right) = a^2 .$$

Ex-2: Find, by double integration, the area of one loop of the lemniscate $r^2 = a^2\cos 2\theta$. [Ans - $\frac{a^2}{2}$] (AKTU-2011).

Ex-3: Find, by double integration, the area of the cardiod $r = a(1+\cos\theta)$. [Ans - $\frac{3}{2}\pi a^2$].

Ex-4: Find by the double integration, the area lying inside the cardioid $r = a(1+\cos\theta)$ & outside the circle $r = a$.

Sol:



Required area

$=$ Area of ABCDA

$= 2 \times$ (Area of ABDA)

$$= 2 \times \iint_{ABDA} r\,dr\,d\theta$$

$$= 2 \int_{\theta=0}^{\pi/2} \int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta = 2 \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= a^2 \int_0^{\pi/2} [(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2 \int_0^{\pi/2} [\cos^2\theta + 2\cos\theta]\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2} + 2\right) = \frac{a^2}{2}(\pi + 8) .$$

MATHEMATICS-I

KAS-103T

Lecture No - 39

Module - IV

[redacted]

Volume by triple integration → (Application of triple integral).

The volume V of a three-dimensional region R is given by

$$\boxed{V = \iiint_R dV = \iiint_R dx\,dy\,dz.}$$

Ex-1 (i) Calculate the volume of the solid bounded by the surface $x=0$, $y=0$, $x+y+z=1$ & $z=0$. (AKTU-2010).

Soln.

Given $x+y+z=1 \Rightarrow z = 1-x-y$.

Put $z=0$, $y=1-x$

& Put $y=0, z=0$, $x=1$.

Hence for the region, here

$0 \le x \le 1$, $0 \le y \le 1-x$, $0 \le z \le 1-x-y$.

Hence required volume

$$V = \iiint_V dV = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\Big[z\Big]_0^{1-x-y} dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}[(1-x)-y]\,dy\,dx.$$

$$= \int_{x=0}^{1}\left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1}\left[(1-x)^2 - \frac{(1-x)^2}{2}\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{1}(1-x)^2\,dx. = \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= \frac{1}{2}\times\left[0+\frac{1}{3}\right] = \frac{1}{6}.$$

(ii) Find the volume of the tetrahedron bdd by the planes $x=0, y=0, z=0$ & $x+y+z=a$

AKTU-(2005) (2008)

$\boxed{\text{Ans } \frac{a^3}{6}}$

Ex-2: find by triple integration, the volume in the +ve octant bounded by the coordinate planes and the plane $x+2y+3z=4$.

[Hint $\int_{x=0}^{4}\int_{y=0}^{\frac{4-x}{2}}\int_{z=0}^{\frac{4-x-2y}{3}} dz\,dy\,dx$] [Ans - $\frac{16}{9}$].

Ex-3: A triangular prism is formed by planes whose equations are $ay=bx$, $y=0$ & $x=a$. find the volume of the prism b/w the planes $z=0$ and surface $z=c+xy$. (AKTU-2010).

Sol: Given $ay=bx$, $y=0$ & $x=a$.

$\Rightarrow$ $x=0$ to a & $y=0$ to $y=\frac{bx}{a}$.

Also given $z=0$ to $z=c+xy$.

Hence the volume of the triangular prism.

$$V=\iiint dV = \int_{x=0}^{a}\int_{y=0}^{\frac{bx}{a}}\int_{z=0}^{c+xy} dz\,dy\,dx.$$

$$= \int_{x=0}^{a}\int_{y=0}^{\frac{bx}{a}}\left[z\right]_0^{c+xy} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{\frac{bx}{a}}(c+xy)\,dy\,dx = \int_{x=0}^{a}\left[cy+x\frac{y^2}{2}\right]_0^{\frac{bx}{a}} dx.$$

$$= \int_{x=0}^{a}\left[\frac{cb}{a}x+\frac{b^2}{2a^2}x^3\right]dx.$$

$$= \frac{cb}{a}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a^2}\left(\frac{x^4}{4}\right)_0^a.$$

$$= \frac{c.b}{a}\cdot\frac{a^2}{2} + \frac{b^2}{2a^2}\cdot\frac{a^4}{4}$$

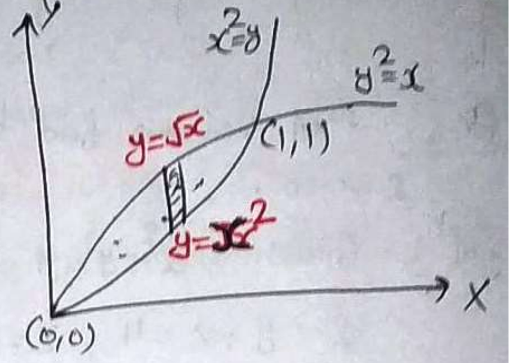
$$= \frac{abc}{2} + \frac{b^2a^2}{8} = \frac{ab}{8}(4c+ab).$$

Ex-3 → find the volume of the cylindrical column standing on the area common to the parabolas $x=y^2$, $y=x^2$ as base and cut-off by the surface $z=12+y-x^2$.

Solⁿ → from figure is clear that $x=0$ to $x=1$

& $y=x^2$ to $y=\sqrt{x}$.

Also $z=0$ to $z=12+y-x^2$.



Hence required volume is

$$V=\iiint dV=\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}\int_{z=0}^{12+y-x^2} dz\,dy\,dx$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}\left[z\right]_0^{12+y-x^2} dy\,dx.$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}(12+y-x^2)\,dy\,dx$$

$$=\int_{x=0}^{1}\left[(12-x^2)\,y+\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx.$$

$$=\int_{x=0}^{1}\left[(12-x^2)(\sqrt{x}-x^2)+\frac{1}{2}(x-x^4)\right]dx.$$

$$=\int_{x=0}^{1}\left(12\sqrt{x}-12x^2-x^{5/2}+x^4+\frac{x}{2}-\frac{x^4}{2}\right]dx.$$

$$=\int_{x=0}^{1}\left[12\sqrt{x}+\frac{x}{2}-12x^2-x^{5/2}+\frac{x^4}{2}\right]dx.$$

$$=\left[12\,\frac{x^{3/2}}{3/2}+\frac{x^2}{4}-12\,\frac{x^3}{3}-\frac{x^{7/2}}{7/2}+\frac{x^5}{10}\right]_0^1$$

$$=8+\frac{1}{4}-4-\frac{2}{7}+\frac{1}{10}=\frac{569}{140}.$$

Ex-4: find the volume of the region bdd by the surface $y=x^2$, $x=y^2$ and the planes $z=0$, $z=3$.

[Hint $\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}\int_{z=0}^{3} dz\,dy\,dx = 1$] [AKTU - 2010, 2019].

Ex-5: find volume bdd by cylinder $x^2+y^2=4$ & planes $y+z=4$ & $z=0$. (AKTU- 2006, 2012).

Sol^n: Given $x^2+y^2=4 \longrightarrow (1)$

& $y+z=4$, $z=0$ $\} \rightarrow (2)$

Put $y=0$ in (1), we get $x^2=4 \Rightarrow x=\pm 2$

& $y^2=4-x^2 \Rightarrow y=\pm\sqrt{4-x^2}$

$\therefore$ $x=-2$ to $x=2$

& $y=-\sqrt{4-x^2}$ to $y=\sqrt{4-x^2}$

Also $z=0$ to $z=4-y$.

Hence required volume,

$$V=\iiint dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{4-y} dz\,dy\,dx.$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[z]_0^{4-y}\,dy\,dx.$$

$$=\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\,dx.$$

$$= 2\int_{x=-2}^{2}\int_{y=0}^{\sqrt{4-x^2}} 4\,dy\,dx.$$

$\left[\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x)=f(x)\right.$

$= 0$ if $f(-x)=-f(x)$

$$= 8\int_{x=-2}^{2}[y]_0^{\sqrt{4-x^2}}\,dx$$

$$= 8\int_{x=-2}^{2}\sqrt{4-x^2}\,dx = 16\int_{x=0}^{2}\sqrt{4-x^2}\,dx$$

$$=16\left[\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{2}\cdot 4\sin^{-1}\frac{x}{2}\right]_0^2 = 16\times 2\times\frac{\pi}{2} = 16\pi.$$

Ex-4→ find the volume of the region bdd by the surface $y = x^2$, $x = y^2$ and the planes $z = 0$, $z = 3$. [AKTU- 2010, 2019]

Hint $\left(\int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{3} dz\, dy\, dx \right)$

Ex-6→ find the volume bounded by the cylinder $x^2+y^2=a^2$ and the planes $y+z=2a$ and $z=0$. [AKTU-2002, 2004, 2006].

Ex-7→ find the volume bdd by xy-plane, the cylinder $x^2+y^2=1$ & plane $x+y+z=3$.

[Hint $\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{3-x-y} dz\, dy\, dx = 3\pi$]

Ex-8→ find the volume enclosed by the cylinder $x^2+y^2=9$ & the planes $x+z=5$ & $z=0$. [Ans 45π]

Ex-9→ find the volume common to the cylinders $x^2+y^2=a^2$ & $x^2+z^2=a^2$.

[Hint $V = \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, dx = \frac{16a^3}{3}$.

Ex-10→ Find the volume bdd by the cylinders $y^2=x$ & $x^2=y$ b/w the planes $z=0$ and $x+y+z=2$ [Ans $\frac{11}{30}$]. (AKTU-2013).

Volume by Dirichlet's Integral→

$$\boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n+1}}$$

Only for first octant

where V is the region bounded by $x \geq 0$, $y \geq 0$, $z \geq 0$ & tetrahedron $x+y+z \leq 1$. (

Ex-1→ find the volume bdd by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. (AKTU - 2012).

Sol→ Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. $\longrightarrow$ ①

Required volume $= \iiint_V dV = \iiint_V dx\, dy\, dz \longrightarrow$ ②

put $\frac{x}{a} = u \Rightarrow x = au \Rightarrow dx = a\,du$

$\frac{y}{b} = v \Rightarrow y = bv \Rightarrow dy = b\,dv$

$\frac{z}{c} = w \Rightarrow z = cw \Rightarrow dz = c\,dw$

Then ① gives $u+v+w \le 1$ & $u \ge 0, v \ge 0, w \ge 0$

Using these values in eqⁿ ②, we get

$$V = \iiint_V a\,du\; b\,dv\; c\,dw$$

$$= abc \iiint_V u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw$$

[Using Dirichlet's integral $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$

$$= abc \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma(1+1+1+1)}$$

$$= abc \frac{\lfloor 0\, \lfloor 0\, \lfloor 0}{\lfloor 3}$$

[$\because \Gamma n = \lfloor n-1$] if n is +ve integer.

$$\Rightarrow \boxed{V = \frac{abc}{6}}.$$

Ex-2: Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (AKTU-2014)

Solⁿ: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \longrightarrow ①$

Required volume $V = 8 \times$ Volume in first octant

$$= 8 \iiint_V dV$$

$$V = 8 \iiint_V dx\, dy\, dz \longrightarrow ②$$

Let $\frac{x^2}{a^2} = u \Rightarrow x = a\sqrt{u}$ then $dx = \frac{a}{2} u^{-1/2} du$

$\frac{y^2}{b^2} = v \Rightarrow y = b\sqrt{v}$ then $dy = \frac{b}{2} v^{-1/2} dv$

$\frac{z^2}{c^2} = w \Rightarrow z = c\sqrt{w}$ then $dz = \frac{c}{2} w^{-1/2} dw$.

then ① becomes, $u+v+w \le 1$

& $u \ge 0, v \ge 0, w \ge 0$. (for first octant).

Using in ②,

$$V = 8\iiint_V \frac{a}{2} u^{-1/2} du \cdot \frac{b}{2} v^{-1/2} dv \cdot \frac{c}{2} w^{-1/2} dw$$

$$= abc \iiint_V u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$$

$$= abc \iiint_V u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1}\, du\, dv\, dw$$

$$= abc \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)}$$

[by Dirichlet's integral $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$.]

$$= abc \frac{\sqrt{\pi}\sqrt{\pi}\sqrt{\pi}}{\Gamma\left(\frac{3}{2}+1\right)}$$

[$\Gamma\frac{1}{2} = \sqrt{\pi}$, $\Gamma(n+1) = n\Gamma n$]

$$= abc \frac{\sqrt{\pi}\sqrt{\pi}\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}$$

$$\Rightarrow V = abc \frac{\sqrt{\pi}\sqrt{\pi}\sqrt{\pi}}{\frac{3}{4}\sqrt{\pi}} = \frac{4abc}{3}\pi.$$

Ex-3: Find the volume of the sphere $x^2+y^2+z^2=a^2$ by triple integration. [Ans $\frac{4}{3}\pi a^3$].

Hint [put $\frac{x^2}{a^2} = u$, $\frac{y^2}{a^2} = v$, $\frac{z^2}{a^2} = w$].

Ex-4: Compute $\iiint_V x^2\, dx\, dy\, dz$ over volume of tetrahedron bdd by $x=0$, $y=0$, $z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$. (AKTU-2018)

Ans $\frac{a^3bc}{60}$.

Ex-5: Evaluate $\iiint x^2yz\, dx\, dy\, dz$ throughout the volume bounded by planes $x=0$, $y=0$, $z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Ans $\frac{a^3b^2c^2}{2520}$. (AKTU-2017)

Solⁿ: Let $I = \iiint x^2yz\, dx\, dy\, dz \rightarrow ①$

Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \rightarrow ②$

put $\frac{x}{a} = u \Rightarrow x = au$ then $dx = a\,du$

$\frac{y}{b} = v \Rightarrow y = bv$ then $dy = b\,dv$

$\frac{z}{c} = w \Rightarrow z = cw$ then $dz = c\,dw$

then ② gives $u+v+w \le 1$ & $u \ge 0, v \ge 0, w \ge 0$.

Now ① becomes,

$$V = \iiint (au)^2 (bv)(cw)\, a\,du\, b\,dv\, c\,dw$$

$$= a^3 b^2 c^2 \iiint u^2 v w \, du\, dv\, dw$$

$$= a^3 b^2 c^2 \iiint u^{3-1} v^{2-1} w^{2-1} \, du\, dv\, dw$$

$$= a^3 b^2 c^2 \frac{\Gamma 3\, \Gamma 2\, \Gamma 2}{\Gamma 3+2+2+1} = a^3 b^2 c^2 \frac{\Gamma 3\, \Gamma 2\, \Gamma 2}{\Gamma 8}$$ [by Dirichlet's integral]

$$= a^3 b^2 c^2 \frac{\lfloor 2 \lfloor 1 \lfloor 1}{\lfloor 7}$$ [$\because \Gamma n = \lfloor n-1$]

$$= a^3 b^2 c^2 \frac{2 \cdot 1}{7 \cdot 5 \cdot 6 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{a^3 b^2 c^2}{2520}$$