* Mathematics - I *
* KAS-103 T *
* LECTURE NO-27 *

* Module - III *
* Differential Calculus - II *

Taylor's Theorem and MacLaurin's Theorem of one Variable -

Let $f(x)$ be a func^n of x. If func^n $f(x+h)$ can be expanded in a convergent series of +ve integral power of h. Then

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \cdots + \frac{h^n}{\lfloor n} f^n(x) + \cdots$$

Deductions :-

① If we replacing a by x, then the expansion about $x=a$ in power of h is

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^n}{\lfloor n} f^n(a) + \cdots$$

② If we replacing h and x each other then the expansion of $f(x)$ in power of x is

$$f(h+x) = f(h) + \frac{x}{\lfloor 1} f'(h) + \frac{x^2}{\lfloor 2} f''(h) + \cdots + \frac{x^n}{\lfloor n} f^n(h) + \cdots$$

③ If we put $a+h = x \Rightarrow h = x-a$, it is more useful form of Taylor's theorem in power of $(x-a)$ about the point $x=a$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n} f^n(a) + \cdots$$

④ If we put $a=0 \Rightarrow h=x$, it is Maclaurin's series in power of x at $x=0$ (fixed point)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^n}{\lfloor n} f^n(0) + \cdots$$

Ex:- find the expansion of a^x.

Sol^n:- Let $f(x) = a^x$.

The expansion of $f(x)$ at a fixed point $x=0$ in power of x. Using Maclaurin's theorem

$$f(x) = f(0) + \frac{x}{\lfloor 1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^n}{\lfloor n} f^n(0) + \cdots \longrightarrow ①$$

$f(x) = a^x$ $\quad\therefore f(0) = a^0 = 1$

$f'(x) = a^x \log a$ $\quad\therefore f'(0) = \log a$

$f''(x) = a^x (\log a)^2$ $\quad\therefore f''(0) = (\log a)^2$

$\vdots$

$f^n(x) = a^x (\log a)^n$ $\quad\therefore f^n(0) = (\log a)^n$

Putting these values in ①, we get

$$f(x) = 1 + x\log a + \frac{x^2}{\lfloor 2}(\log a)^2 + \cdots + \frac{x^n}{\lfloor n}(\log a)^n + \cdots$$

H.W. Ex- Prove that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots$ ② e^{x+h} ③ $\frac{1}{x+h}$.

Ex- Expand $2x^3 + 7x^2 + x - 6$ in power of $(x-2)$ [MTU-2012]

Solⁿ- Let $F(x) = 2x^3 + 7x^2 + x - 6$.

Using Taylor's theorem in power of $(x-2)$ about the point $x=2$ is.

$$F\left(\frac{x-2}{h} + \frac{2}{a}\right) = F(2) + \frac{(x-2)}{\lfloor 1} f'(2) + \frac{(x-2)^2}{\lfloor 2} f''(2) + \cdots \longrightarrow ①$$

we have $F(x) = 2x^3 + 7x^2 + x - 6$ then $F(2) = 40$

$F'(x) = 6x^2 + 14x + 1$ then $F'(2) = 53$

$F''(x) = 12x + 14$ then $F''(2) = 38$

$F'''(x) = 12$ then $F'''(2) = 12$

$F^{iv}(x) = 0$ then $f^{iv}(2) = 0$.

Put in ①, we get

$$F(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3. \text{ Ans}$$

T.T Q- Expand $\tan x$ in power of $(x - \frac{\pi}{4})$ upto four terms.

Q- Expand $\sin x$ in ascending powers of $(x - \frac{\pi}{2})$.

②
$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \left(\frac{x}{1+x}\right)^2 \frac{1}{\lfloor 2} f''(x)$$
$$- \left(\frac{x}{1+x}\right)^3 \frac{1}{\lfloor 3} f'''(x)$$

Ex:- Evaluate $\sqrt{25.15}$ Using Taylor's theorem.

Solⁿ:- Let $F(x) = \sqrt{x}$ where $x = 25.15 = 25 + 0.15$
$= a + h$.
where $a = 25$, $h = 0.15$

Using Taylor's theorem in power of 'h' about $x = a$ is

$$F(a+h) = F(a) + h\,f'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots \quad \rightarrow ①$$

Now $F(x) = \sqrt{x}$ then $f(25) = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$ then $f'(25) = \frac{1}{10}$

$F''(x) = -\frac{1}{4x^{3/2}}$ then $F''(25) = \frac{1}{1000}$

Put in ①, we get

$$\sqrt{25.15} = 5 + \frac{0.15}{10} + \frac{(0.15)^2}{1000} + \cdots = 5.01478 \text{ (Approx.)}$$

* Numericals :-

① Obtain $\tan^{-1}x$ in power of $(x-1)$

② find the expansion of e^x by M. Theorem.

③ Obtain the series for $\log(1+x)$ and find series for $\log\left(\frac{1+x}{1-x}\right)$ and determine the value of $\log\frac{11}{9}$ upto five places of decimal. (UPTU-2012)

④ Expand $e^{2x}\sin x$ in ascending power of x upto x^5. (UPTU - 2014)

⑤ Show that $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\right)$ (UPTU-2015)

⑥ MT Expand by Maclaurin's theorem $\frac{e^x}{e^x+1}$ as far as x^3.

⑦ Expand $\log\sin x$ in power of $(x-a)$. TT

$f(x) = \log(1+x) \quad \therefore f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} \quad \therefore f'(0) = 1$

$f''(x) = (-1)(1+x)^{-2} \quad \therefore f''(0) = -1$

$f'''(x) = (-1)(-2)(1+x)^{-3} \quad \therefore f'''(0) = 2$

$f^{iv}(x) = (-1)(-2)(-3)(1+x)^{-4} \quad \therefore f^{iv}(0) = -6$

and so on.

$\therefore$ by M.T $f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$

Changing x into $-x$, we get

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots$$

Now $\log(1+x) - \log(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right]$

Put $x = \frac{1}{10}$,

$$\log\left(\frac{11}{9}\right) = 2\left[\frac{1}{10} + \frac{1}{3}\left(\frac{1}{10}\right)^3 + \frac{1}{5}\left(\frac{1}{10}\right)^5 + \frac{1}{7}\left(\frac{1}{10}\right)^7 \dots\right]$$

$$= 0.20067 \text{ (Approx)}$$

* Expand $\sin^{-1}x$ upto four terms in power of x.
* Expand $\tan^{-1}x$ upto four terms.

Solⁿ: $f(x) = \tan^{-1}x \quad \therefore f(0) = 0$

then $f'(x) = \frac{1}{1+x^2}$

$= (1+x^2)^{-1}$

$= 1 - x^2 + x^4 - x^6 + \dots \quad \therefore f'(0) = 1$

$f''(x) = -2x + 4x^3 - 6x^5 + \dots \quad f''(0) = 0$

$f'''(0) = -2$

$f^{iv}(0) = 0 \quad f^{v}(0) = 24$

$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

* Show that $\tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} \dots\right)$ (UPTU-2015)

* $e^{ax}\sin bx = bx + abx^2 + \frac{b(3a^2-b^2)}{\lfloor 3}x^3 + \dots + \frac{(a^2+b^2)^{n/2}}{\lfloor n}x^n \sin(n\tan^{-1}\frac{b}{a})$

* $f(x) = e^{2x}\sin x, \quad f(0) = 0$ (2014)

$f'(x) = e^{2x}\cos x + 2e^{2x}\sin x$

$f'(x) = e^{2x}\cos x + 2f(x) \quad \therefore f'(0) = 1$

$f''(x) = -e^{2x}\sin x + 2e^{2x}\cos x + 2f'(x)$

$\Rightarrow f''(x) = -f(x) + 2[f'(x) - 2f(x)] + 2f'(x)$

$\Rightarrow f''(x) = 4f'(x) - 5f(x) \quad \therefore f''(0) = 4$

$f'''(x) = 4f''(x) - 5f'(x) \quad \therefore f'''(0) = 11$

Q: $e^{x}\cos x = 1 + x - \frac{2x^3}{\lfloor 3} - \frac{2^2x^4}{\lfloor 4} - \frac{2^2x^5}{\lfloor 5}$

* $\sec x$
* $\log \sec x$

Taylor's Series.
or
Taylor's Theorem for func^n of two variables →

If $f(x,y)$ and all its partial derivatives upto nth order are finite and continuous for all points (x,y) where

$$a \le x \le a+h \quad , \quad b \le y \le b+K$$

Then

$$\boxed{\begin{aligned} f(a+h, b+K) = f(a,b) &+ \frac{1}{\underline{|1}}\left[h f_x(a,b) + K f_y(a,b)\right] \\ &+ \frac{1}{\underline{|2}}\left[h^2 f_{xx}(a,b) + 2hK f_{xy}(a,b) + K^2 f_{yy}(a,b)\right] \\ &+ \cdots \end{aligned}}$$

Type-1

Note-1 → Put $h=x$, $K=y$ & $a=0, b=0$, we get (Taylor Theorem about $(0,0)$)

$$\boxed{\begin{aligned} f(x,y) = f(0,0) &+ \left[x f_x(0,0) + y f_y(0,0)\right] \\ &+ \frac{1}{\underline{|2}}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] \\ &+ \frac{1}{\underline{|3}}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) \right. \\ &\left. + y^3 f_{yyy}(0,0)\right] + \cdots \end{aligned}}$$

It is called Maclaurin's Theorem for func^n of two variables.

Ex-1 → Show that $e^y \log(1+x) = x + xy - \frac{x^2}{2}$ approximately.

Sol^n → Given $f(x,y) = e^y \log(1+x) \longrightarrow ①$

Then by Taylor's Theorem about $(0,0)$ i.e by Maclaurin's Theorem we have

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{\underline{|2}}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots \longrightarrow ②$$

Now from ①,

* $f(x,y) = e^y \log(1+x)$ then $\boxed{f(0,0) = 0}$
* $f_x(x,y) = \dfrac{e^y}{1+x}$ then $\boxed{f_x(0,0) = 1}$
* $f_y(x,y) = e^y \log(1+x)$ then $\boxed{f_y(0,0) = 0.}$

* $f_{xx}(x,y) = \frac{-e^y}{(1+x)^2}$ then $\boxed{f_{xx}(0,0) = -1}$

* $f_{xy}(x,y) = \frac{e^y}{1+x}$ then $\boxed{f_{xy}(0,0) = 1}$

* $f_{yy}(x,y) = e^y \log(1+x)$ then $\boxed{f_{yy}(0,0) = 0}$.

Using these values in eqn ②, we get

$$f(x,y) = x + xy - \frac{x^2}{2} + \cdots\cdots$$

$$\Rightarrow \boxed{f(x,y) = x + xy - \frac{x^2}{2}} \text{ (Approximately)}$$

Ex-2: Find first six terms of the expansions of func^n $e^x \log(1+y)$ in a Taylor's Series about the point $(0,0)$. (AKTU -2015)

Ans: $e^x \log(1+y) = y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \cdots$

Ex-3: Expand $e^x \sin y$ in powers of x & y at $(0,0)$ as far as terms of third degree. (AKTU-2010) (2008).

Ans $e^x \sin y = y + xy + \frac{x^2y}{2} - \frac{1}{2}y^3 + \cdots\cdots$

Ex-4: Expand $e^{ax}\sin by$ in power of x and y as far as the terms of third degree. (AKTU-2011).

Type-II:

Note: Put $a+h = x$ i.e $h = x-a$
& $b+k = y$ i.e $k = y-b$ in Taylor's Theorem we get

$$\boxed{\begin{aligned} f(x,y) = f(a,b) &+ [(x-a) f_x(a,b) + (y-b) f_y(a,b)] \\ &+ \frac{1}{\lfloor 2}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) \\ &\quad + (y-b)^2 f_{yy}(a,b)] + \cdots \end{aligned}}$$

This is called Taylor's Theorem in power of $(x-a)$ & $(y-b)$ about the point (a,b).

Ex-1: Express the func^n $f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$ as Taylor's series expansion about the point $(1,2)$.

(AKTU - 2009, 2017, 2018).

Sol^n: 1: Given $f(x,y) = x^2 + 3y^2 - 9x - 9y + 26 \longrightarrow (1)$.

Now Taylor's Theorem in the power of $(x-a)$ & $(y-b)$ i.e The Taylor's Theorem about the point (a,b) is

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{\lfloor 2}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots \longrightarrow (2)$$

Put $a=1$ & $b=2$ in eq^n (2), we get

Taylor's series of $f(x,y)$ about the point $(1,2)$,

$$f(x,y) = f(1,2) + [(x-1) f_x(1,2) + (y-2) f_y(1,2)] + \frac{1}{\lfloor 2}[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2) f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)] + \cdots \longrightarrow (3)$$

Now from (1),

$f(x,y) = x^2 + 3y^2 - 9x - 9y + 26$ then $\boxed{f(1,2) = 12}$

$f_x(x,y) = 2x - 9$ then $\boxed{f_x(1,2) = -7}$

$f_y(x,y) = 6y - 9$ then $\boxed{f_y(1,2) = 3}$

$f_{xx}(x,y) = 2$ then $\boxed{f_{xx}(1,2) = 2}$

$f_{xy}(x,y) = 0$ then $\boxed{f_{xy}(1,2) = 0}$

$f_{yy}(x,y) = 6$ then $\boxed{f_{yy}(1,2) = 6}$

Using these values in eq^n (3), we get

$$f(x,y) = 12 + (x-1)(-7) + (y-2)\cdot 3 + \frac{1}{\lfloor 2}[(x-1)^2 \cdot 2 + 2(x-1)(y-2)\cdot 0 + (y-2)^2 \cdot 6] + \cdots$$

$$\Rightarrow f(x,y) = 12 - 7(x-1) + 3(y-2) + (x-1)^2 + 3(y-2)^2 + \cdots$$

Ex-2: Expand e^{xy} at $(1,1)$. (Ans- $e\{1 + (x-1) + (y-1) + \frac{1}{\lfloor 2}\{(x-1)^2 + 4(x-1)(y-1) + (y-1)^2\} + \cdots\}$.

Ex-3: Expand $e^x \cos y$ at $(1, \frac{\pi}{4})$. (AKTU-2008, 2010).

[Ans: $\frac{e}{\sqrt{2}}\{1 + (x-1) - (y-\frac{\pi}{4}) + \frac{(x-1)^2}{2} - (x-1)(y-\frac{\pi}{4}) - \frac{1}{2}(y-\frac{\pi}{4})^2 + \cdots$

Ex-4: Expand $\sin xy$ in powers of $(x-1)$ & $(y-\frac{\pi}{2})$ up to second degree terms. (AKTU-2011).

Ans $1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\frac{\pi}{2}) - \frac{1}{2}(y-\frac{\pi}{2})^2 + \cdots$

Ex-5: Expand $f(x,y) = y^x$ about $(1,1)$ upto second degree terms and hence evaluate $(1.02)^{1.03}$. (AKTU-2013).

Ans $y^x = 1 + (y-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \cdots$

& $(1.02)^{1.03} = 1.020606$.

Ex-6: Expand x^y in powers of $(x-1)$ & $(y-1)$ up to third degree terms and hence evaluate $(1.1)^{1.02}$. (AKTU - 2010).

Ans $1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \cdots$ & $(1.1)^{1.02} = 1.1021$.

Ex-7: Expand $f(x,y) = e^x \tan^{-1} y$ in powers of $(x-1)$ & $(y-1)$ up to two terms of degree 2. (AKTU-2013).

Ans $e^x \tan^{-1} y = \frac{\pi}{4} e + \frac{e}{2}\left[(x-1)\frac{\pi}{2} + (y-1)\right]$

$+ \frac{e}{4}\left[(x-1)^2 \frac{\pi}{2} + 2(x-1)(y-1) - (y-1)^2\right] + \cdots$

Ex-8: Expand $(x^2 y + \sin y + e^x)$ in powers of $(x-1)$ & $(y-\pi)$. (AKTU-2014)

Ans: $\pi + e + (x-1)(2\pi + e) + \frac{1}{2}(x-1)^2(2\pi + e) + 2(x-1)(y-\pi) + \cdots$

Ex-9: Expand $\tan^{-1}\frac{y}{x}$ in the neighbourhood of $(1,1)$ upto second degree terms. Hence compute $f(1.1, 0.9)$ approximately. (AKTU-2007, 2014).

Solⁿ 9: Given $f(x,y) = \tan^{-1}\frac{y}{x} \longrightarrow (1)$

By Taylor's theorem about the point (a,b) i.e in powers of $(x-a)$ & $(y-b)$, we have

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)]$$
$$+ \frac{1}{\lfloor 2}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots \longrightarrow (2)$$

Put $a=1, b=1$ in eqⁿ (2), we get Taylor's Theorem or Series about the point $(1,1)$ or in the neighbourhood of $(1,1)$,

$$f(x,y) = f(1,1) + [(x-1) f_x(1,1) + (y-1) f_y(1,1)]$$
$$+ \frac{1}{\lfloor 2}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \cdots \longrightarrow (3)$$

from eq^n ①, we get

⊛ $f(x,y) = \tan^{-1}\frac{y}{x}$ then $\boxed{f(1,1) = \tan^{-1}1 = \frac{\pi}{4}}$

⊛ $f_x(x,y) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$ then $\boxed{f_x(1,1) = -\frac{1}{2}}$

⊛ $f_y(x,y) = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$ then $\boxed{f_y(1,1) = \frac{1}{2}}$.

⊛ $f_{xx}(x,y) = \frac{-y\cdot(-1)}{(x^2+y^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$ then $\boxed{f_{xx}(1,1) = \frac{1}{2}}$

⊛ $f_{xy}(x,y) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$ then $\boxed{f_{xy}(1,1) = 0}$.

⊛ $f_{yy}(x,y) = x\,\frac{(-1)}{(x^2+y^2)^2} \cdot 2y = -\frac{2xy}{(x^2+y^2)^2}$ then $\boxed{f_{yy}(1,1) = -\frac{1}{2}}$

Using those values in eq^n ③, we get

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\cdot\frac{1}{2}\right]$$
$$+ \frac{1}{\lfloor 2}\left[(x-1)^2\cdot\frac{1}{2} + 2(x-1)(y-1)\cdot 0 + (y-1)^2\cdot\left(-\frac{1}{2}\right)\right] + \cdots$$

$$\Rightarrow \boxed{\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \cdots} \rightarrow ④$$

Put $x = 1.1$ & $y = 0.9$ in eq^n ④, we get

$$f(1.1, 0.9) = \tan^{-1}\frac{1.1}{0.9} = \frac{\pi}{4} - \frac{1}{2}(1.1-1) + \frac{1}{2}(0.9-1)$$
$$+ \frac{1}{4}(1.1-1)^2 - \frac{1}{4}(0.9-1)^2 + \cdots$$

$= 0.6857$ (Approximately).

* Lecture No - 29 *

[redacted]

* Maxima & Minima of a funcⁿ of several variables →

* Extreme (Maximum or Minimum) Values →

*1→ A funcⁿ $f(x,y)$ is said to have a maximum value at $x=a, y=b$, if there exists a small neighbourhood of (a,b) such that

$$\boxed{f(a,b) > f(a+h, b+k)}$$

*2→ A funcⁿ $f(x,y)$ is said to have a minimum value for $x=a, y=b$, if there exists a small nbd of (a,b) such that

$$\boxed{f(a,b) < f(a+h, b+k)}$$

Note→ ① Maximum or Minimum Value ⇒ Extreme value.

② A point where $f(x,y)$ is neither max nor min called saddle point.

③ A point where the funcⁿ $f(x,y)$ is either max or min is called stationary point.

Working Rule→ Given $z=f(x,y)$

① find $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$, $r=\frac{\partial^2 z}{\partial x^2}$, $s=\frac{\partial^2 z}{\partial x \partial y}$, $t=\frac{\partial^2 z}{\partial y^2}$.

② Put $\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$, for max or min and find stationary point (a,b).

③ find r, s, t at the points (a,b).

④
(i) If $rt-s^2>0$ & $r<0$ then $f(x,y)$ has maximum value at (a,b)

(ii) If $rt-s^2>0$ & $r>0$ then $f(x,y)$ has minimum value at (a,b).

(iii) If $rt-s^2<0$ then $f(x,y)$ is neither max nor min at (a,b).

(iv) If $rt-s^2=0$ then the case is doubtful.

Ex-1: Discuss the maxima and minima of

$$x^2+y^2+6x+12.$$

or find the extreme value of $f(x,y)=x^2+y^2+6x+12$. (AKTU-2011)

Sol: Given $z=x^2+y^2+6x+12 \longrightarrow (1)$

then
$$p=\frac{\partial z}{\partial x}=2x+6$$
$$q=\frac{\partial z}{\partial y}=2y$$
$$r=\frac{\partial^2 z}{\partial x^2}=2$$
$$s=\frac{\partial^2 z}{\partial x\partial y}=0$$
$$t=\frac{\partial^2 z}{\partial y^2}=0$$

for maxima or minima, put

$$\frac{\partial f}{\partial x}=0 \Rightarrow 2x+6=0 \Rightarrow x=-3$$
$$\frac{\partial f}{\partial y}=0 \Rightarrow 2y=0 \Rightarrow y=0.$$

Then $(-3,0)$ is the stationary point for the curve.

At $(-3,0)$ → $r=2,\ s=0,\ t=2.$

$\therefore rt-s^2=2\cdot2-0^2=4>0$ & $r=2>0$

then $f(x,y)$ is min at point $(-3,0)$ & its extreme or minimum value is

$$f(-3,0)=(-3)^2+0^2+6\times(-3)+12=3. \quad (\text{from } (1))$$

Ex-2: Find the stationary point of the curve x^3+y^3-3axy. Examine also for maximum or minimum values. (AKTU-2017).

Sol: Let $f(x,y)=x^3+y^3-3axy \longrightarrow (1)$

then $p=\frac{\partial f}{\partial x}=3x^2-3ay$, $q=\frac{\partial f}{\partial y}=3y^2-3ax$, $r=\frac{\partial^2 f}{\partial x^2}=6x$

$s=\frac{\partial^2 f}{\partial x\partial y}=-3a$, $t=\frac{\partial^2 f}{\partial y^2}=6y$.

for max or min, put $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$

$\Rightarrow 3x^2-3ay=0$, $3y^2-3ax=0.$

$\Rightarrow x^2-ay=0 \longrightarrow (2)$, $y^2-ax=0 \longrightarrow (3)$

from (2), $y=\frac{x^2}{a}$. Using in (3), we get

$$\frac{x^4}{a^2}-ax=0 \Rightarrow x^4-a^3x=0 \Rightarrow x(x^3-a^3)=0$$
$$\Rightarrow x=0,\ x=a.$$

Then from (2) $y=0,\ y=a.$

Hence stationary points are $(0,0)$ & (a,a).

At (0,0): we have

$r = 0$, $s = -3a$, $t = 0$

$\therefore rt - s^2 = 0 - (-3a)^2 = -9a^2 < 0 \Rightarrow rt - s^2 < 0$

Then func^n $f(x,y)$ has no extreme value at $(0,0)$.

At (a,a): we get $r = 6a$, $s = -3a$, $t = 6a$

$\therefore rt - s^2 = 6a \times 6a - (-3a)^2 = 36a^2 - 9a^2 = 27a^2 > 0$.

Now $r = 6a$

(i) if $a < 0$ then $r < 0 \Rightarrow f(x,y)$ has max value at (a,a)

if $a < 0$. Max Value is $f(a,a) = (-a)^3 + (-a)^3 + 3a(-a)(-a)$
$= 5a^3$.

(ii) if $a > 0 \Rightarrow r > 0$ then $f(x,y)$ has min value at (a,a).

& Min Value is $f(a,a) = a^3 + a^3 - 3a^3 = -a^3$.

Ex-3: Show that the minimum value of $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$. (AKTU-2013).

(Ans- Point (0,0), (a,a) & Min at (a,a) and Value is $3a^2$)

Ex-4: find the extreme value of

$x^3 + y^3 - 63(x+y) + 12xy$. (AKTU-2011).

Sol^n: Let $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy \longrightarrow (1)$

Then $p = \frac{\partial f}{\partial x} = 3x^2 - 63 + 12y$

$q = \frac{\partial f}{\partial y} = 3y^2 - 63 + 12x$

$r = \frac{\partial^2 f}{\partial x^2} = 6x$, $s = \frac{\partial^2 f}{\partial x \partial y} = 12$, $t = \frac{\partial^2 f}{\partial y^2} = 6y$.

For max or min put $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 3x^2 - 63 + 12y = 0$, $3y^2 - 63 + 12x = 0$

$\Rightarrow x^2 + 4y = 21 \rightarrow (2)$, $y^2 + 4x = 21 \rightarrow (3)$

On subtracting (2) & (3),

$(x^2 - y^2) + 4(y - x) = 0$

$\Rightarrow (x-y)(x+y) - 4(x-y) = 0$

$\Rightarrow (x-y)(x+y-4) = 0$

$\Rightarrow x - y = 0$, $x + y - 4 = 0$

If $x = y$, then (2) gives

$x^2 + 4x = 21 \Rightarrow x^2 + 4x - 21 = 0 \Rightarrow (x-3)(x+7) = 0 \Rightarrow x = 3, -$

then $y = 3, -7$.

$\therefore$ Points are $(3,3)$, $(-7,-7)$.

Using $x+y-4=0$ in ②, we get

$$x^2+4(4-x)=21 \Rightarrow x^2-4x-5=0$$
$$\Rightarrow (x+1)(x-5)=0 \Rightarrow x=-1,5.$$

Put $x=-1$ in ②, $y=5$

Put $x=5$ in ②, $y=-1$.

$\therefore$ Points are $(-1,5)$, $(5,-1)$.

Hence stationary point are $(3,3)$, $(-7,-7)$, $(-1,5)$ & $(5,-1)$.

At $(3,3)$ → $r=18$, $s=12$, $t=18$

$\therefore rt-s^2=18\times18-12^2=180>0$ & $r=18>0$.

Then funcⁿ has min at $(3,3)$ & min value is

$$f(3,3)=3^3+3^2-63(3+3)+12\times3\times3$$
$$=-216.$$

At $(-7,-7)$, $(-1,5)$, $(5,-1)$ $\boxed{\text{Try yourself.}}$

Ex-5 → find the stationary point of

$$f(x,y)=5x^2+10y^2+12xy-4x-6y+1$$

& discuss for max or min. (AKTU-2012).

Ex-6 → find extreme val of $x^3y^2(6-x-y)$.

Ans [Points are $(0,0)$, $(3,2)$, $(0,4)$, $(6,0)$.

Ex-7 → In a plane triangle ABC, find the maximum value of $\cos A\cos B\cos C$. (AKTU-2010)

Solⁿ → We have $\cos A\cos B\cos C = \cos A\cos B\cos[\pi-(A+B)]$

$$=-\cos A\cos B\cos(A+B) \quad [\because A+B+C=\pi]$$

Let $f(A,B)=-\cos A\cos B\cos(A+B)$

Then $\dfrac{\partial f}{\partial A}=-\cos B[-\cos A\sin(A+B)-\sin A\cos(A+B)]$

$$=\cos B\sin(2A+B)$$

$\dfrac{\partial f}{\partial B}=-\cos A[-\sin B\cos(A+B)-\cos B\sin(A+B)]$

$$=\cos A\sin(A+2B)$$

$$r = \frac{\partial^2 f}{\partial A^2} = 2\cos B \cos(2A+B) \longrightarrow (a)$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = \cos A \cos(A+2B) - \sin A \sin(A+2B)$$

$$= \cos(2A+2B) \longrightarrow (b)$$

$$t = \frac{\partial^2 f}{\partial B^2} = 2\cos A \cos(A+2B) \rightarrow (c)$$

For max or min, put

$$\frac{\partial f}{\partial A} = 0 \qquad , \frac{\partial f}{\partial B} = 0$$

$$\Rightarrow \cos B \sin(2A+B) = 0 \longrightarrow (1) \qquad , \cos A \sin(A+2B) = 0 \longrightarrow (2)$$

[If $\cos B = 0$ then $B = \frac{\pi}{2}$ therefore (2) gives

$$\cos A \cdot \sin(A+\pi) = 0 \Rightarrow -\cos A \sin A = 0$$

$$\Rightarrow \cos A = 0 \quad , \sin A = 0$$

$$\Rightarrow A = \frac{\pi}{2} \quad , A = 0, \pi .$$

which is not possible as $A+B+C=0$.]

$$\Rightarrow \cos B \neq 0 \quad , \cos A \neq 0 .$$

Then $\sin(2A+B) = 0 \Rightarrow 2A+B = \pi$

$\sin(A+2B) = 0 \Rightarrow A+2B = \pi$

On solving $A = \frac{\pi}{3}$, $B = \frac{\pi}{3}$.

Hence stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ or $A = B = \frac{\pi}{3}$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right) \rightarrow$

$$r = 2\cos\frac{\pi}{3} \cos\left(2\frac{\pi}{3} + \frac{\pi}{3}\right) = 2\cos\frac{\pi}{3} \cos\pi = -1 \quad (\text{from } (a))$$

$$s = \cos\left(2\frac{\pi}{3} + 2\frac{\pi}{3}\right) = \cos\frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2} \quad (\text{from } b)$$

$$t = 2\cos\frac{\pi}{3} \cos\pi = -1 .$$

$$\therefore rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0 \quad \text{Also } r = -1 < 0 .$$

$\Rightarrow f(A, B)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Maximum value is

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} \cos\frac{\pi}{3} \cos\frac{2\pi}{3} = -\frac{1}{2}\frac{1}{2} \cdot \frac{-1}{2} = \frac{1}{8} .$$

<u>Ex-8</u> → Examine for minimum and maximum values of $\sin x + \sin y + \sin(x+y)$.

Example 4. *Examine for minimum and maximum values: sin x + sin y + sin (x + y).*

Sol. Here $f(x, y) = \sin x + \sin y + \sin (x + y)$

$$f_x = \cos x + \cos (x + y)$$

$$f_y = \cos y + \cos (x + y)$$

$$r = f_{xx} = -\sin x - \sin (x + y)$$

$$s = f_{xy} = -\sin (x + y)$$

$$t = f_{yy} = -\sin y - \sin (x + y)$$

Now, $f_x = 0$ and $f_y = 0$

$\Rightarrow$ $\cos x + \cos (x + y) = 0$ ...(1) and $\cos y + \cos (x + y) = 0$ ...(2)

Subtracting equation (2) from (1),

$$\cos x - \cos y = 0 \text{ or } \cos x = \cos y \quad \therefore \quad x = y$$

From (1), $\cos x + \cos 2x = 0$

or $$\cos 2x = -\cos x = \cos (\pi - x)$$

or $$2x = \pi - x \quad \therefore \quad x = \frac{\pi}{3}$$

$\therefore$ $x = y = \dfrac{\pi}{3}$ is a stationary point.

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $\quad r = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3},\ s = \dfrac{\sqrt{3}}{2},\ t = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore$ $$rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

Also $r < 0$

$\therefore$ $f(x, y)$ has a maximum value at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

Maximum value $= f\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right) = \sin \dfrac{\pi}{3} + \sin \dfrac{\pi}{3} + \sin \dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{2}$.

Example 5. *In a plane triangle ABC, find the maximum value of cos A cos B cos C.*

Lagrange's Method of Multipliers →

To find the maximum or minimum values of a funcⁿ of three (or more) variables, when the variables are not independent but are connected by some given relation. Then we use to find the extreme value by Lagrange's Method of Multipliers.

Working Rule →

① Consider the funcⁿ $u = f(x, y, z)$ (which is to be extremized) where x, y, z are connected by the relation $\phi(x, y, z) = 0$.

② find $du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \longrightarrow$

& $\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \longrightarrow (2)$

③ Put $du = 0$ for maxima or minima, we get

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \longrightarrow (1)$$

④ Multiplying eqⁿ ② by λ & adding to eqⁿ ①, we get Lagrange's identity

$$\boxed{\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right) = 0}$$

⑤ Lagrange's identity will be hold if

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad &\Rightarrow \quad -\lambda = \frac{\partial f}{\partial x}\Big/\frac{\partial \phi}{\partial x} \\ \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad &\Rightarrow \quad -\lambda = \partial f/\partial y\Big/\frac{\partial \phi}{\partial y} \\ \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad &\Rightarrow \quad -\lambda = \partial f/\partial z\Big/\frac{\partial \phi}{\partial z} \end{aligned}\right\} \longrightarrow (3)$$

Solve eqⁿ ③ & find the value of x, y & z.

⑥ Put the values of x, y & z in given eqⁿ, we get extreme value of u.

Ex-1 → find the minimum value of $x^2+y^2+z^2$ when $xyz = a^3$.

Solⁿ → Let $u = x^2+y^2+z^2 \longrightarrow (1)$ (because u has to minimized)

$$\phi = xyz - a^3 = 0 \longrightarrow (2)$$

Now by total differentiation, we get

$$du = 2x\,dx + 2y\,dy + 2z\,dz$$

$$yz\,dx + xz\,dy + xy\,dz = 0 \longrightarrow (4)$$

for max or min put $du = 0 \Rightarrow x\,dx + y\,dy + z\,dz = 0 \longrightarrow (3)$

Now by Lagrange's identity, we have

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\Rightarrow (x + \lambda yz)\,dx + (y + \lambda xz)\,dy + (z + \lambda xy)\,dz = 0$$

$$\Rightarrow \left.\begin{aligned} x + \lambda yz = 0 &\Rightarrow -\lambda = \frac{x}{yz} \\ y + \lambda xz = 0 &\Rightarrow -\lambda = y/xz \\ z + \lambda xy = 0 &\Rightarrow -\lambda = \frac{z}{xy} \end{aligned}\right\} \rightarrow (5)$$

from (5),

$$\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} \Rightarrow \frac{x^2}{xyz} = \frac{y^2}{xyz} = \frac{z^2}{xyz}$$

$$\Rightarrow x^2 = y^2 = z^2 \Rightarrow \boxed{x = y = z}$$

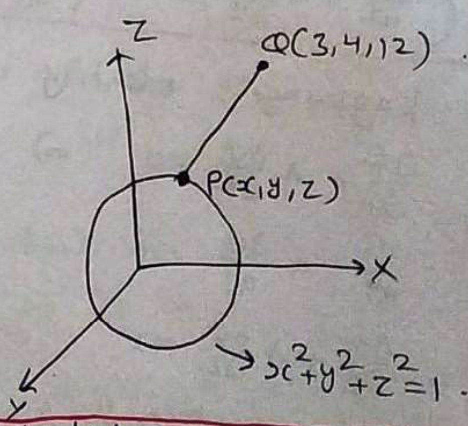
Put $x = y = z$ in eq^n (2), we get $x^3 = a^3 \Rightarrow \boxed{x = a}$

$$\therefore \boxed{x = a,\ y = a,\ z = a} \text{ (Stationary point)}$$

Hence minimum value of u is $\boxed{u_{min} = a^2 + a^2 + a^2 = 3a^2}$

Ex-2: Find the maximum and minimum distances from the point $(3, 4, 12)$ to the sphere $x^2 + y^2 + z^2 = 1$. (AKTU-2007, 2010).

Sol^n: Let $P(x, y, z)$ be any point on the sphere & $Q(3, 4, 12)$.



Then

$$PQ = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

$$\Rightarrow (PQ)^2 = (x-3)^2 + (y-4)^2 + (z-12)^2.$$

Let $u = (x-3)^2 + (y-4)^2 + (z-12)^2 \longrightarrow (1)$

(As max or min distance have to find)

& $\phi = x^2 + y^2 + z^2 - 1 = 0 \longrightarrow (2)$

Now by total diff., we get

$$du = 2(x-3)\,dx + 2(y-4)\,dy + 2(z-12)\,dz$$

$$2x\,dx + 2y\,dy + 2z\,dz = 0$$

For max or min, put $du = 0$, then

$$(x-3)\,dx + (y-4)\,dy + (z-12)\,dz = 0 \longrightarrow (3)$$

$$x\,dx + y\,dy + z\,dz = 0 \longrightarrow (4)$$

Now by Lagrange's identity, we have $\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)=0$.

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)dz=0$$

$$\Rightarrow [(x-3)+\lambda x]dx+[(y-4)+\lambda y]dy+[(z-12)+\lambda z]dz=0$$

$$\left.\begin{aligned} &\Rightarrow (x-3)+\lambda x=0 \Rightarrow -\lambda=\frac{x-3}{x}=1-\frac{3}{x}\\ &(y-4)+\lambda y=0 \Rightarrow -\lambda=\frac{y-4}{y}=1-\frac{4}{y}\\ &(z-12)+\lambda z=0 \Rightarrow -\lambda=\frac{z-12}{z}=1-\frac{12}{z}\end{aligned}\right\}\to (5)$$

From eqⁿ (5), $1-\frac{3}{x}=1-\frac{4}{y}=1-\frac{12}{z}$

$$\Rightarrow -\frac{3}{x}=-\frac{4}{y}=-\frac{12}{z} \Rightarrow \frac{3}{x}=\frac{4}{y}=\frac{12}{z}=K\text{(say)}.$$

then $x=\frac{3}{K},\ y=\frac{4}{K},\ z=\frac{12}{K} \longrightarrow (6)$

Using these values in eqⁿ (2), we get

$$\frac{3^2}{K^2}+\frac{4^2}{K^2}+\frac{12^2}{K^2}=1 \Rightarrow K^2=169 \Rightarrow \boxed{K=\pm 13}$$

Hence (6) gives,

$$x=\pm\frac{3}{13},\ y=\pm\frac{4}{13},\ z=\pm\frac{12}{13}.$$

∴ Points are
$$P_1\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right) \ \& \ P_2\left(-\frac{3}{13},-\frac{4}{13},-\frac{12}{13}\right).$$

Hence $QP_1=\sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}$

$=12$

& $QP_2=\sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}=14.$

Hence minimum distance of Q from P is 12
& maximum distance is 14.

Ex-3 Find the shortest and the longest distances from the point (1, 2, -1) to the sphere $x^2+y^2+z^2=24$. (AKTU - 2009, 2015, 2018).
[Ans - $\sqrt{6}$, $3\sqrt{6}$].

Ex-4 Divide a number into three parts so that the product of first, square of second and cube of third is maximum. (AKTU - 2004, 2017)

Ex-5 Divide 24 into 3-parts, such that the continued product of the first, square of the second and the cube of the third may be maximum. (AKTU - 2012, 2014).

Sol^n-5 Let three parts of 24 are x, y, z. Then

$$x+y+z = 24$$

s.t. xy^2z^3 may be max.

Let $u = xy^2z^3 \longrightarrow (1)$ & $\phi = x+y+z-24=0 \longrightarrow (2)$

Now by total differentiation, (1) & (2) gives

$$du = y^2z^3dx + 2xyz^3dy + 3xy^2z^2dz$$

$$\& \quad dx+dy+dz=0.$$

For max or min put $du=0$.

$$y^2z^3dx + 2xyz^3dy + 3xy^2z^2dz = 0 \longrightarrow (3)$$

$$dx+dy+dz=0 \longrightarrow (4)$$

Now by Lagrange's identity, we have

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial\phi}{\partial z}\right)dz = 0$$

$$\Rightarrow (y^2z^3 + \lambda)dx + (2xyz^3+\lambda)dy + (3xy^2z^2+\lambda)dz = 0.$$

$$\left.\begin{aligned}
\Rightarrow\quad & y^2z^3 + \lambda = 0 \Rightarrow -\lambda = y^2z^3 \\
& 2xyz^3 + \lambda = 0 \Rightarrow -\lambda = 2xyz^3 \\
& 3xy^2z^2 + \lambda = 0 \Rightarrow -\lambda = 3xy^2z^2
\end{aligned}\right\} \rightarrow (5)$$

from (5), $y^2z^3 = 2xyz^3 = 3xy^2z^2$

$$\Rightarrow \frac{y^2z^3}{xy^2z^3} = \frac{2xyz^3}{xy^2z^3} = \frac{3xy^2z^2}{xy^2z^3}$$

$$\Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{3}{z} = K \text{ (say)}.$$

$$\Rightarrow x = \frac{1}{K},\ y = \frac{2}{K},\ z = \frac{3}{K} \longrightarrow (6)$$

Using in eq^n (2), we get

$$\frac{1}{K} + \frac{2}{K} + \frac{3}{K} = 24 \Rightarrow \frac{6}{K} = 24 \Rightarrow \boxed{K = \frac{1}{4}}$$

Hence from (6), $\boxed{x=4,\ y=8,\ z=12}$

& $u_{max} = 4\cdot 8^2\cdot 12^3$ Ans

Ex-6 Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

i) Box is open at the top.

ii) Box is closed. (AKTU- 2014, 2016, 2018).

Sol^n+6) Let x, y, z be the dimensions of the rectangular box so that its volume is $V = xyz \longrightarrow (1)$

(Capacity)

& Total surface area of the box is

$S = hxy + 2yz + 2zx$ = Constant (given) $\longrightarrow (2)$

Here $h=1$, if the box is open at the top.

& $h=2$, if the box is closed.

Let $V = xyz \longrightarrow (3)$

& $\phi = hxy + 2yz + 2zx - S = 0 \longrightarrow (4)$

By total diff,

$$dV = yz\,dx + xz\,dy + xy\,dz$$

& $(hy+2z)\,dx + (hx+2z)\,dy + (2y+2x)\,dz = 0$.

For max or min, put $dV=0$.

$$yz\,dx + xz\,dy + xy\,dz = 0 \longrightarrow (5)$$

& $(hy+2z)\,dx + (hx+2z)\,dy + (2y+2x)\,dz = 0 \longrightarrow (6)$

Now by Lagrange's identity,

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right)dz = 0$$

$$\Rightarrow [yz + \lambda(hy+2z)]\,dx + [xz + \lambda(hx+2z)]\,dy + [xy + \lambda(2y+2x)]\,dz = 0$$

$$\left.\begin{aligned} &\Rightarrow yz + \lambda(hy+2z) = 0 \Rightarrow -\lambda = \frac{yz}{hy+2z} \\ &xz + \lambda(hx+2z) = 0 \Rightarrow -\lambda = \frac{xz}{hx+2z} \\ &xy + \lambda(2x+2y) = 0 \Rightarrow -\lambda = \frac{xy}{2x+2y} \end{aligned}\right\} \longrightarrow (7)$$

from (7),
$$\frac{yz}{hy+2z} = \frac{xz}{hx+2z} = \frac{xy}{2x+2y}$$

$$\Rightarrow \frac{xyz}{hxy+2zx} = \frac{xyz}{hxy+2yz} = \frac{xyz}{2xz+2yz}$$

$$\Rightarrow \underbrace{hxy + 2zx}_{(i)} = \underbrace{hxy + 2yz}_{(ii)} = \underbrace{2xz + 2yz}_{(iii)}$$

Taking (i) & (iii), $hxy + 2zx = 2xz + 2yz$

$$\Rightarrow hxy = 2yz \Rightarrow hx = 2z$$

Taking (ii) & (iii), $hxy + 2yz = 2xz + 2yz$

$$\Rightarrow \boxed{hy = 2z}$$

Hence $\boxed{nx = ny = 2z} \longrightarrow (8)$

(i) when box is open at the top, $n=1$,

$$x = y = 2z$$

$\therefore$ from (2), $S = x^2 + x^2 + x^2$ $\quad [\because S = nxy + 2yz + 2zx]$

$$\Rightarrow x^2 = \frac{S}{3} \Rightarrow x = \sqrt{\frac{S}{3}}$$

Then $y = \sqrt{\frac{S}{3}}$, $2z = \sqrt{\frac{S}{3}}$ i.e $z = \frac{1}{2}\sqrt{\frac{S}{3}}$

(ii) when box is closed, $n=2$,

then (8) gives

$$x = y = z.$$

Using in (2), $S = 2x^2 + 2x^2 + 2x^2$

$$\Rightarrow x^2 = \frac{S}{6} \Rightarrow x = \sqrt{\frac{S}{6}}$$

Then $y = \sqrt{\frac{S}{6}}$, $z = \sqrt{\frac{S}{6}}$

Ex-7→ A rectangular box, which is open at the top, has a capacity of 32 cc. Determine, using Lagrange method of multipliers, the dimensions of the box such that the least material is required for the construction of the box. (AKTU- 2014, 2015)

Note→ Consider $u = xy + 2xz + 2yz$ & $\phi = V = xyz = 32$ } as (least or min material is asked in question).

Ex-8→ A rectangular box, which is open at the top, has a capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is required for the construction of the box. Use L.M. of Multiplier to obtain the soln. (AKTU-2011).

Ex-9→ The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.

Note→ Let $u = xyz$, $\phi = x+y+z-k$.

Ex-10→ Find the minimum value of $x^2+y^2+z^2$, given that $ax+by+cz=p$. (AKTU-2012).

Example 17. *Find the minimum value of* $x^2 + y^2 + z^2$, *given that* $ax + by + cz = p$.

(U.K.T.U. 2012)

Sol. Let $$u = x^2 + y^2 + z^2 \quad ...(1)$$

where $$\phi(x, y, z) = ax + by + cz - p = 0 \quad ...(2)$$

Consider Lagrange's function, $F(x, y, z) = (x^2 + y^2 + z^2) + \lambda(ax + by + cz - p)$

For stationary values, $dF = 0$

$$\Rightarrow \quad (2x + \lambda a)dx + (2y + \lambda b)dy + (2z + \lambda c)dz = 0$$

$$\Rightarrow \quad 2x + \lambda a = 0 \quad ...(3)$$

$$2y + \lambda b = 0 \quad ...(4)$$

$$2z + \lambda c = 0 \quad ...(5)$$

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

or $$2u + \lambda p = 0 \quad | \text{ Using (1) and (2)}$$

$$\therefore \quad \lambda = -\frac{2u}{p}$$

From (3), (4) and (5), $$x = \frac{au}{p}, y = \frac{bu}{p}, z = \frac{cu}{p}$$

$\therefore$ From (1), $$u = \frac{(a^2 + b^2 + c^2)u^2}{p^2}$$

or $$u = \frac{p^2}{a^2 + b^2 + c^2}.$$

This is the **maximum or minimum** value of u. Now u is the square of the distance of any point ...

O

A(0, 0, 0)

Example 11. *Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.*

Sol. Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular solid and let R be the radius of the sphere.

Then, $x^2 + y^2 + z^2 = R^2$ ...(1)

Volume $V = 8xyz$ ...(2)

Consider Lagrange's function

$$F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \{8yz + \lambda(2x)\}\, dx + \{8xz + \lambda(2y)\}\, dy + \{8xy + \lambda(2z)\}\, dz = 0$$

$$\Rightarrow \quad 8yz + 2\lambda x = 0 \quad ...(3)$$

$$8zx + 2\lambda y = 0 \quad ...(4)$$

$$8xy + 2\lambda z = 0 \quad ...(5)$$

From (3), $2\lambda x^2 = -8xyz$

From (4), $2\lambda y^2 = -8xyz$

From (5), $2\lambda z^2 = -8xyz$

$\therefore$ $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$

or $x^2 = y^2 = z^2$ or $x = y = z$.

Hence, rectangular solid is a cube.

Example 12. *Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$ (AKTU-2019)(U.K.T.U. 2010)

Sol. Let (x, y, z) be a vertex of the parallelopiped then it lies on the given ellipsoid

$$\therefore \quad \phi(x, y, z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad ...(1)$$

Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid. ...(2)

$\therefore$ Volume $V = 8xyz$

Consider Lagrange's function

$$F(x, y, z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \left\{8yz + \lambda\left(\frac{2x}{a^2}\right)\right\} dx + \left\{8zx + \lambda\left(\frac{2y}{b^2}\right)\right\} dy + \left\{8xy + \lambda\left(\frac{2z}{c^2}\right)\right\} dz = 0 \quad ...(3)$$

$$\Rightarrow \quad 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0$$

$$8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad \text{...(4)}$$

$$8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad \text{...(5)}$$

Multiplying (3), (4), (5) by x, y, z respectively and adding, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$\Rightarrow \quad 24xyz + 2\lambda = 0 \quad \Rightarrow \quad \lambda = -12xyz$$

From (3), $\quad 8yz - 12xyz\left(\frac{2x}{a^2}\right) = 0$

$$\Rightarrow \quad 1 - \frac{3x^2}{a^2} = 0 \quad \Rightarrow \quad x = \frac{a}{\sqrt{3}}$$

Similarly from (4) and (5), we get $y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$.

$\therefore$ Volume of the largest rectangular parallelopiped $= 8xyz = \frac{8abc}{3\sqrt{3}}$.

Example 13. *The temperature T at any point (x, y, z) in space is $T = 400\, xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$.*

Sol. Consider Lagrange's function

$$F(x, y, z) = 400\, xyz^2 + \lambda(x^2 + y^2 + z^2 - 1) \quad \text{...(1)}$$

For stationary values,

$$dF = 0$$

$$\Rightarrow \quad [400yz^2 + \lambda(2x)]\, dx + [400\, xz^2 + \lambda(2y)]\, dy + [800\, xyz + \lambda(2z)]\, dz = 0$$

$$\Rightarrow \quad 400yz^2 + 2\,\lambda x = 0 \quad \text{...(2)}$$

$$400xz^2 + 2\,\lambda y = 0 \quad \text{...(3)}$$

$$800xyz + 2\,\lambda z = 0 \quad \text{...(4)}$$

Multiplying (2) by x, (3) by y and (4) by z and adding, we get

$$1600xyz^2 + 2\,\lambda(x^2 + y^2 + z^2) = 0$$

$$\Rightarrow \quad \lambda = -800xyz^2 \quad \text{...(5)} \quad \Big| \because \; x^2 + y^2 + z^2 = 1$$

From (2), $\quad 400yz^2 - 1600x^2yz^2 = 0 \quad \Big|$ Using (5)

$$\Rightarrow \quad x = \pm\frac{1}{2}$$

Similarly, $\quad y = \pm\frac{1}{2}$

From (4) $\quad 800xyz - 1600xyz^3 = 0$

$$\Rightarrow \quad 1 - 2z^2 = 0$$

$$\Rightarrow \quad z = \pm\frac{1}{\sqrt{2}}$$

Putting values of x, y, z in T, we get

$(\text{Max})\, T = 400 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 50$

MATHEMATICS-I
KAS-103T
* Lecture No - 31 *

Module-III
* Diff. Calculus-II *
* [redacted]

Jacobians:

Jacobian is a functional determinant, usefull in transformation of variables from cartesian to polar, cylindrical and spherical polar coordinates in multiple integrals.

Case-I: If $u = f_1(x, y)$ & $v = f_2(x, y)$ are the functions of two independent variables, then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$$

is called Jacobian of u, v w.r.t x, y and it is denoted by $J(u, v)$.

Ex-1: If $x = u(1+v)$, $y = v(1+u)$ then find $\frac{\partial(x,y)}{\partial(u,v)}$. (AKTU-2010)

Sol: Given $x = u(1+v) \longrightarrow (1)$

$y = v(1+u) \rightarrow (2)$

Now, we know that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv$$

$$= 1+u+v.$$

Ex-2: If $x = e^{v}\sec u$, $y = e^{v}\tan u$ then evaluate $\frac{\partial(x,y)}{\partial(u,v)}$.

Ans $-e^{2v}\sec u$. (AKTU-2010)

Ex-3: (i) If $u = x(1-y)$, $v = xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$. (Ans. x)

(ii) If $u = x^2 - y^2$, $v = 2xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$. (Ans- $\frac{1}{4(x^2+y^2)}$).

Case-II: If $u = f(x,y,z)$, $v = \phi(x,y,z)$ & $w = \psi(x,y,z)$ are functions of x, y & z. Then the Jacobian is

$$\boxed{J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}}$$

Ex-1) If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ the show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4 .$$ (AKTU-2015, 2018).

Sol) Given $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_1 x_3}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

Now
$$J(y_1, y_2, y_3) = \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_1} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_1 x_3 & x_1 x_2 \\ x_2 x_3 & -x_1 x_3 & x_1 x_2 \\ x_3 x_2 & x_3 x_1 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{(x_2 x_3)(x_1 x_3)(x_1 x_2)}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1)$$

$$= 0 + 2 + 2 = 4.$$ (Hence proved)

Ex-2) If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta$ and find $\frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$. (AKTU-2009)

Ex-3) Calculate the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ of the following
$u = x + 2y + z$, $v = x + 2y + 3z$, $w = 2x + 3y + 5z$. [Ans - 2]

Ex-4) If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$, then compute the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$. [Ans $-(x-y)(y-z)(z-x)$.

⊛ Properties of Jacobians →

① If u, v are functions of r, s where r, s are functions of x, y, then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}} \quad \text{(Chain Rule).}$$

Ex-1 → If $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$ then calculate the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$. (AKTU-2013, 2015).

Solⁿ → By chain Rule, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} \longrightarrow ①$$

Given $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$

Now
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\frac{\sqrt{w}}{\sqrt{v}} & \frac{1}{2}\frac{\sqrt{v}}{\sqrt{w}} \\ \frac{1}{2}\frac{\sqrt{w}}{\sqrt{u}} & 0 & \frac{1}{2}\frac{\sqrt{u}}{\sqrt{w}} \\ \frac{1}{2}\frac{\sqrt{v}}{\sqrt{u}} & \frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\sqrt{\frac{w}{v}}\sqrt{\frac{u}{w}}\sqrt{\frac{v}{u}} + \frac{1}{8}\sqrt{\frac{v}{w}}\sqrt{\frac{w}{u}}\sqrt{\frac{u}{v}}$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}. \longrightarrow ②$$

Also we have,

$u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

Now
$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \cancel{\sin\theta\cos\phi \quad r\cos\theta\cos\phi \quad -r\sin\theta\sin\phi}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \left\{ \cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} + 0 \right\}$$

$$= r^2\sin\theta \left\{ \cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi) + \sin\theta(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi) \right\}$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta \rightarrow (3)$$

Using (2) & (3) in (1), we get

$$\therefore \boxed{\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{1}{4} \times r^2\sin\theta = \frac{1}{4}r^2\sin\theta}.$$

Ex-2: If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$, find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$. **(AKTU-2012)**. Ans $\frac{-1}{2(x-y)(y-z)(z-x)}$

[Hint - find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ then $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$].

Ex-3: If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$. [Ans - r]

Ex-4: If $u = x^2-y^2$, $v = 2xy$ & $x = r\cos\theta$, $y = r\sin\theta$

show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Ex-5: If $x = v^2+w^2$, $y = w^2+u^2$, $z = u^2+v^2$

then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$. **(AKTU-2017)**.

[Hint find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ & $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$ & then proove.].

Property-2: If u, v are functions of x & y and x, y are the funcn of u & v then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1 \Rightarrow J_1 \cdot J_2 = 1}.$$

Ex-14 If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$

Also prove that $JJ' = 1$. (AKTU-2014).

Sol^n:

Given

$$x = r\cos\theta \;,\; y = r\sin\theta \longrightarrow (1)$$

then $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

From (1),

$$\left.\begin{aligned} r^2 = x^2 + y^2 \quad & \text{i.e } r = \sqrt{x^2+y^2} \\ \& \tan\theta = \frac{y}{x} \quad & \text{i.e } \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\} \rightarrow (2)$$

Now $J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{(x^2+y^2)} & \frac{x}{(x^2+y^2)} \end{vmatrix}$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{(x^2+y^2)}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

from (2)

Hence $\boxed{J \cdot J' = r \cdot \frac{1}{r} = 1}$ Hence proved.

(*) **Property-3:** →

If
$$y_1 = f(x_1)$$
$$y_2 = f(x_1, x_2)$$
$$y_3 = f(x_1, x_2, x_3)$$
$$\vdots$$
$$y_n = f(x_1, x_2, \ldots x_n)$$
Then
$$\frac{\partial(y_1, y_2, y_3 \ldots y_n)}{\partial(x_1, x_2, x_3 \ldots x_n)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdots \frac{\partial y_n}{\partial x_n}.$$

Ex-1, If
$$y_1 = 1 - x_1$$
$$y_2 = x_1(1-x_2)$$
$$y_3 = x_1 x_2 (1-x_3)$$
$$y_4 = x_1 x_2 x_3 (1-x_4)$$
$$\vdots$$
$$y_n = x_1 x_2 x_3 \ldots x_{n-1}(1-x_n)$$

then show that
$$\frac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

Sol^n 1 → Given $y_1 = 1-x_1$

$$y_2 = x_1(1-x_2)$$
$$y_3 = x_1x_2(1-x_3)$$
$$\vdots$$
$$y_n = x_1x_2x_3 \dots x_{n-1}(1-x_n)$$

Then

$$\frac{\partial(y_1, y_2, \dots y_n)}{\partial(x_1, x_2, \dots x_n)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3} \dots \frac{\partial y_n}{\partial x_n} \quad \text{(by Property III)}.$$

$$= (-1)(-x_1)(-x_1x_2)(-x_1x_2x_3) \dots (-x_1x_2 \dots x_{n-1})$$
$$= (-1)^n x_1^{n-1} x_2^{n-2} \dots x_{n-1}.$$

Ex-2 → If $y_1 = 1-x_1$, $y_2 = x_1(1-x_2)$, $y_3 = x_1x_2(1-x_3)$

Find $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$. (AKTU-2013).

Sol^n → Given $y_1 = 1-x_1$

$$y_2 = x_1(1-x_2)$$
$$y_3 = x_1x_2(1-x_3)$$

Now by Property 3 of Jacobian,

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3}$$
$$= (-1)(-x_1)(-x_1x_2)$$
$$= -x_1^2 x_2.$$

Ex-3 → If $y_1 = \cos x_1$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_1 \sin x_2 \cos x_3$ then show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = -\sin^3 x_1 \sin^2 x_2 \sin x_3.$$

Property - IV → Let u, v, w are the functions of three variable x, y, z.

V. Imp If $J(u, v, w) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$.

Then u, v, w are dependent or functional dependence.

or u, v, w are not independent.

or u, v, w are related to each other.

Ex-1: If $u = x+2y+z$, $v = x-2y+3z$, $w = 2xy - xz + 4yz - 2z^2$ then show that u, v, w are not independent. Find the relation b/w u, v & w. (AKTU-2017).

Sol^n: Given $u = x+2y+z \longrightarrow (1)$

$v = x-2y+3z \longrightarrow (2)$

$w = 2xy - xz + 4yz - 2z^2 \longrightarrow (3)$

Now
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & 2x-4y+6z & -x+2y-3z \end{vmatrix}$$ by $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - C_1$

$$= -4(-x+2y-3z) - 2(2x-4y+6z)$$
$$= 4x - 8y + 12z - 4x + 8y - 12z = 0$$

Hence u, v & w are not independent.

$\Rightarrow$ u, v, w are functionally related.

Now from (1) & (2)

$$u+v = 2x+4z \quad , \quad u-v = 4y-2z.$$

Multiplying, we get

$$(u+v)(u-v) = (2x+4z)(4y-2z)$$
$$= 4(2xy + 4yz - zx - 2z^2)$$
$$= 4w \quad (\text{from } (3))$$

$\Rightarrow$ $\boxed{u^2 - v^2 = 4w}$ Required relation b/w u, v, w.

Ex-2: If $u = \frac{x+y}{z}$, $v = \frac{y+z}{x}$, $w = \frac{y(x+y+z)}{xz}$, then show that u, v, w are not independent and find the relation b/w them. (AKTU-2010).

Ans $[\, uv = w+1 \,]$.

Ex-3: Use the Jacobian to prove that the funct^n $u = x+y-z$, $v = x-y+z$, $w = x^2+y^2+z^2-2yz$ are not independent of one another. Find relation.

Ex-3: Use the Jacobian to prove that the funcⁿ $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ & $w = x + y + z$ are not independent of one another. find the relation b/w them. (AKTU-2010).

Ans $\boxed{w^2 = v + 2u}$

Ex-4: Show that $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ are not independent. find the relation b/w them. (AKTU-2014).

Ans $\boxed{2u^2 = v - w}$

Ex-5: If $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x(x + 2y - z)$, then show that they are not independent and prove that $u^2 - v^2 = 8w$

Ex-6: If $u = \tan^{-1}x + \tan^{-1}y$ & $v = \frac{x+y}{1-xy}$, then prove that u & v are related to each other & find relation b/w them.

Ans $v = \tan^{-1}u$.

Ex-7: Are the functions

$u = \frac{x-y}{x+z}$, $v = \frac{x+z}{y+z}$ functionally dependent ?. If so, find the relation b/w them. (AKTU-2012) [Ans $v = \frac{1}{1-u}$.

Ex-8: If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$ then prove that u, v, w are not independent, find relation. [Ans - $2w = u(3v - u^2)$.

Ex-9: Show that the functions:

$u = x + y + z$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$

$w = x^3 + y^3 + z^3 - 3xyz$ (AKTU-2014, 2018)

are functionally related. find the relation b/w them.

Solⁿ: Given $u = x + y + z \longrightarrow (1)$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \longrightarrow (2)$

$w = x^3 + y^3 + z^3 - 3xyz \longrightarrow (3)$

Now
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2x - 2y - 2z & 2y - 2x - 2z & 2z - 2y - 2x \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2x-2y-2z & 4y-4x & -4x+4z \\ 3x^2-3yz & 3(y^2-x^2)+3z(y-x) & 3(z^2-x^2)+3y(z-x) \end{vmatrix} \quad \text{by } C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$$

$$= \begin{vmatrix} 4y-4x & -4x+4z \\ 3(y^2-x^2) + 3z(y-x) & 3(z^2-x^2) + 3y(z-x) \end{vmatrix}$$

$$= 4 \times 3 \begin{vmatrix} y-x & -x+z \\ (y-x)(x+y+z) & (z-x)(x+y+z) \end{vmatrix}$$

$$= 12(x+y+z) \begin{vmatrix} y-x & z-x \\ y-x & z-x \end{vmatrix} = 12(x+y+z) \cdot 0 = 0$$

Hence u, v, w are not independent.

$\cancel{= 12(x+y+z)[yz - xy - xy]}$

Now from (2)

$$v = x^2+y^2+z^2-2xy-2yz-2zx$$

$$= (x^2+y^2+z^2+2xy+2yz+2zx) - 4xy - 4yz - 4zx$$

$$= (x+y+z)^2 - 4(xy+yz+zx)$$

$$\Rightarrow v = u^2 - 4(xy+yz+zx) \quad (\text{from } 1)$$

$$\Rightarrow xy+yz+zx = \frac{u^2-v}{4} \longrightarrow (4)$$

From (3), $w = x^3+y^3+z^3-3xyz$

$$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$= u[(x^2+y^2+z^2-2xy-2yz-2zx) + (xy+yz+zx)]$$

$$= u\left[v + \frac{u^2-v}{4}\right]$$

$$\boxed{w = u\left(\frac{u^2+3v}{4}\right)}$$

MATHEMATICS-I
KAS-103T
* Lecture - No. 32 *

Module - III
* Diff. Calculus - II *
[redacted]

Implicit function:-

A function or relation, in which the dependent variable is not isolated on one side of the eqn is called implicit func^n.

Ex:- $(x+y)^2 = a^2$ or $x^2+y^2+2xy = a^2$.

In the eq^n, y can not be separated in one side.

(*) Jacobians of implicit func^n:-

Let $u_1, u_2, \ldots, u_n$ are implicit functions of $x_1, x_2, x_3 \ldots x_n$.

& let $f_1(u_1, u_2, u_3, \ldots u_n, x_1, x_2 \ldots, x_n) = 0$

$f_2(u_1, u_2, u_3, \ldots u_n, x_1, x_2 \ldots, x_n) = 0$

$\vdots$

$f_n(u_1, u_2, u_3, \ldots u_n, x_1, x_2 \ldots, x_n) = 0$.

Then
$$\boxed{J(u_1, u_2, \ldots, u_n) = \frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \frac{\dfrac{\partial(f_1, f_2, \ldots f_n)}{\partial(x_1, x_2, \ldots x_n)}}{\dfrac{\partial(f_1, f_2, \ldots f_n)}{\partial(u_1, u_2, \ldots, u_n)}}}$$

Ex-1:- If $x+y+z = u$, $y+z = uv$, $z = uvw$ then show that
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v. \quad \text{(AKTU - 2016)}.$$

Sol^n:- Let $f_1(u,v,w,x,y,z) = x+y+z-u = 0$

$f_2(u,v,w,x,y,z) = y+z-uv = 0$

$f_3(u,v,w,x,y,z) = z-uvw = 0$.

Now for three func^n,
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} \longrightarrow (1)$$

Now
$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$$
$$= -u^2 v.$$

& $\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$

Using these values in ①, we get

$$\therefore \frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{-u^2 v}{1} = u^2 v.$$

Ex-2→ If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ then show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u - v)}. \quad \text{(AKTU – 2007)}$$

Ex-3→ If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$

then show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}$.

Ex-4→ If $u_1 = x_1 + x_2 + x_3$, $u_1^2 u_2 = x_1 + x_2$ & $u_1^3 u_3 = x_3$, find the value of Jacobian $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$. [Ans u_1^{-5}].

Ex-5→ If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1 u_2 = x_2 + x_3 + x_4$

$u_1 u_2 u_3 = x_3 + x_4$, $u_1 u_2 u_3 u_4 = x_4$ then

show that $\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3 u_2^2 u_3$. (AKTU - 2013).

Ex-6→ If u, v, w are the roots of the cubic eqn

(i) $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ, then find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

[AKTU-2016]

(ii) If u, v, w are the roots of the eqn

$(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$, then find $\frac{\partial(u, v, w)}{\partial(a, b, c)}$. (AKTU-2015

Soln→ 6(i) Given $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

$$\Rightarrow 3\lambda^3 - 3\lambda^2(x + y + z) + 3\lambda(x^2 + y^2 + z^2) - (x^3 + y^3 + z^3) = 0 \to ①$$

We know that if u, v, w are the roots of

$ax^3 + bx^2 + cx + d = 0$ then

$u + v + w = -\frac{b}{a}$, $uv + vw + wu = \frac{c}{a}$, $uvw = -\frac{d}{a}$.

Now u, v, w are the roots of eqn ① then

$$u+v+w = \frac{3(x+y+z)}{3}, \quad uv+vw+wu = \frac{3(x^2+y^2+z^2)}{3}$$

$$\& \; uvw = \frac{x^3+y^3+z^3}{3}.$$

Let $f_1 \equiv u+v+w-x-y-z=0$

$f_2 = uv+vw+wu-x^2-y^2-z^2=0$

$f_3 = uvw - \frac{1}{3}(x^3+y^3+z^3)=0$

For three func^n, we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} \longrightarrow (1)$$

Now $$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \quad \text{by } C_2 \to C_2 - C_1, \; C_3 \to C_3 - C_1$$

$$= -2\left[(y-x)(z^2-x^2) - (z-x)(y^2-x^2)\right]$$

$$= -2\left[(y-x)(z-x)(z+x) - (z-x)(y-x)(y+x)\right]$$

$$= -2(y-x)(z-x)\left[z+x-y-x\right]$$

$$= 2(x-y)(z-x)(z-y) = -2(x-y)(y-z)(z-x).$$

Also, $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \quad \text{by } C_2 \to C_2 - C_1, \; C_3 \to C_3 - C_1$$

$$= (u-v)v(u-w) - (u-w)w(u-v)$$

$$= (u-v)(u-w)(v-w) = -(u-v)(v-w)(w-u)$$

Using these values in ① , we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{[-2(x-y)(y-z)(z-x)]}{[-(u-v)(v-w)(w-u)]}$$

$$= -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

① *** Approximation of Errors *** * By Anuj Kumar *

* If the variables are related to several other variables by functional relationship, then it is possible to estimate the percentage change (error) in one variable by the given percentage change in other variable.

Let $Z = f(x, y) \rightarrow ①$.

If $\delta x, \delta y$ are the small increment in x & y respectively and δz, the corresponding increment in z, then

$$z + \delta z = f(x+\delta x, y+\delta y) \rightarrow ②$$

Subtracting eqn ① from ②, we get

$$\delta z = f(x+\delta x, y+\delta y) - f(x, y)$$

$$= \left(f(x,y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \cdots\right) - f(x,y)$$

[By Taylor's theorem].

$$\Rightarrow \boxed{\delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}} \text{ (Approximately)} \rightarrow ③$$

If δx and δy are the small changes or (errors) in x & y respectively then δz an approximate change (or error) in z. Replace $\delta x, \delta y, \delta z$ by dx, dy & dz respectively in ③, we get

$$\boxed{dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

Note→ If δx is the error in x, then $\boxed{\frac{\delta x}{x} = \text{relative error}}$

& $\boxed{\frac{\delta x}{x} \times 100 = \text{percentage error}}$.

Working Process→

Step ① Taking logarithm of the given funcn (If possible)

② Differentiate by using total differentiation.

③ Multiply by 100 and put given values of changes or error.

Q-1→ The period of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum (%) error in T due to possible errors upto 1% in l and 2.5% in g. [UPTU-2014]

Soln→ Given $T = 2\pi\sqrt{\frac{l}{g}}$

Taking log, $\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$.

By total differentiation, we get

$$\frac{1}{T}\delta T = \frac{1}{2}\cdot\frac{1}{l}\delta l - \frac{1}{2}\frac{1}{g}\delta g$$

$$\Rightarrow \frac{\delta T}{T} \times 100 = \frac{1}{2}\left[\frac{\delta l}{l}\times 100 - \frac{\delta g}{g}\times 100\right] = \frac{1}{2}\left[1 \pm 2.5\right]$$

$\therefore$ Maximum error in $T = \frac{1}{2}(1+2.5) = \frac{3.5}{2} = 1.75\%$.

Q-2→ The time T of a complete oscillation of a simple pendulum of length 'L' is governed by the eqn $T = 2\pi\sqrt{\frac{L}{g}}$, g is constant. Find the approximate error in the calculate value of T corresponding to an error of 2% in the value of L.

Soln→ We have

$$T = 2\pi\sqrt{\frac{L}{g}},$$

Taking log,

$$\log T = \log 2\pi + \frac{1}{2}\log L - \frac{1}{2}\log g$$

Then by Total differentiation,

$$\frac{\delta T}{T} = \frac{1}{2}\frac{\delta L}{L} \quad [\because g \text{ is constant}].$$

$$\Rightarrow \frac{\delta T}{T}\times 100 = \frac{1}{2}\frac{\delta L}{L}\times 100 = \frac{1}{2}\times 2 = 1.$$

$\therefore$ Approximate error in $T = 1\%$.

(2)

Q.3 If period T of a simple pendulum $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum % error in T due to possible error of 2% in l and 5% in g. [Ans → 3.5%]

Q.4 The power 'p' required to propel a steamer of length 'l' at a speed 'u' is given by $P = \lambda u^3 l^3$ where λ is constant. If u is increased by 3% and l is decreased by 1%, find the corresponding increase in 'p'. [UKTU-2012].

Soln Given $P = \lambda u^3 l^3 \rightarrow (1)$

Taking logarithm both side, we get $\log P = \log\lambda + 3\log u + 3\log l$.

By Total differention, $\frac{\delta P}{P} = 3\frac{\delta u}{u} + 3\frac{\delta l}{l}$

$\Rightarrow \frac{\delta P}{P}\times 100 = 3\frac{\delta u}{u}\times 100 + 3\frac{\delta l}{l}\times 100$

$\Rightarrow \frac{\delta P}{P}\times 100 = 3\times 3 + 3\times(-1)$

$= 6$.

[Given $\frac{\delta u}{u}\times 100 = 3\%$

$\frac{\delta l}{l}\times 100 = -1\%$.]

[-ve sign for decrease]

∴ Increase in 'P' is 6%.

Q.5 The power P required to propel a ship of length l moving with the velocity V is given by $P = KV^3 l^2$. Find the percentage error increase in power, if velocity increase 3% and length increase is 4%. [Ans – 17%].

Q.6 Find the percentage error in measuring the volume of a rectangular box when the error of 1% is made in measuring each side. (AKTU-2017)

Soln The volume of a rectangular box is

V = length × width × height

$\Rightarrow V = lwh \rightarrow (1)$ Taking log both side,

$\log V = \log l + \log w + \log h$

By total differentiation, we get

$\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{\delta w}{w} + \frac{\delta h}{h}$

$\Rightarrow \frac{\delta V}{V}\times 100 = \frac{\delta l}{l}\times 100 + \frac{\delta w}{w}\times 100 + \frac{\delta h}{h}\times 100$ [Given $\frac{\delta l}{l}\times 100 = \frac{\delta w}{w}\times 100 = \frac{\delta h}{h}\times 100 = 1\%$]

$= 1 + 1 + 1 = 3$.

∴ Error in volume of rectangular box = 3%.

Q.7 Find the percentage error in the area of a rectangle when an error of +1% is made in measuring its length & breadth. [$A = l\times b$] [Ans → 2%].

Q.8 Find the percentage error in the area of an ellipse when an error of +1 percent is made in measuring the major and minor axes. [UPTU-2011].

Soln If x and y are semi-major and semi-minor axes of the ellipse then its area is $A = \pi xy$.

Taking logarithm both side, $\log A = \log\pi + \log x + \log y$.

On diff., $\frac{\delta A}{A} = 0 + \frac{\delta x}{x} + \frac{\delta y}{y}$.

$\Rightarrow \frac{\delta A}{A}\times 100 = \frac{\delta x}{x}\times 100 + \frac{\delta y}{y}\times 100 = 1 + 1 = 2$.

∴ Error in the area = 2%.

(*) Calculate the error in R if E = RI and possible errors in E and I are 20% and 10% respectively. [Ans – 10%]

(3) **Q.9** If the base radius and height of a cone are measured as 4 and 8 inches with a possible error of 0.04 and 0.08 inches respectively, calculate the % error in calculating volume of the cone. [UKTU-2012].

Sol^n Volume of cone $V = \frac{1}{3}\pi r^2 h$. Taking log both side, we get

$$\log V = \log\frac{1}{3} + \log\pi + 2\log r + \log h.$$

By total diff.,
$$\frac{\delta V}{V} = 2\frac{\delta r}{r} + \frac{\delta h}{h}$$

$$\Rightarrow \frac{\delta V}{V} = 2\times\frac{0.04}{4} + \frac{0.08}{8} = .03.$$

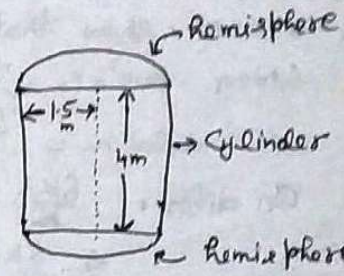
$\therefore$ Percentage error in volume $= \cancel{.03\times 100}\ \frac{\delta V}{V}\times 100 = .03\times 100 = 3\%$.

Q.10 A ballon is in the form of right circular cylinder of radius 1.5m and length 4m and is surmounted by hemispherical ends. If the radius in increased by 0.01 m and length by 0.05m, find the percentage change in the volume of ballon. [AKTU-2014, 2018, 2019].

Sol^n Given radius of cylinder $r = 1.5$m.

Hight (length) of cylinder $h = 4$m.

$\therefore$ Volume of the ballon V



$= $ Volume of cylinder $+$ Volume of two hemisphere

$$= \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$$

$$\Rightarrow V = \pi r^2 h + \frac{4}{3}\pi r^3 \longrightarrow (1)$$

Then $\delta V = \pi.2r\delta r\, h + \pi r^2\, \delta h + \frac{4}{3}\pi.3r^2\delta r$.

$$= \pi r\,[2h\,\delta r + r\,\delta h + 4r\,\delta r] \longrightarrow (2)$$

$$\therefore \frac{\delta V}{V} = \frac{\pi r\,[2h\,\delta r + r\,\delta h + 4r\,\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3} = \frac{2h\,\delta r + r\,\delta h + 4r\,\delta r}{rh + \frac{4}{3}r^2}.$$

$$= \frac{2\times 4\times 0.01 + 1.5\times 0.05 + 4\times 1.5\times 0.01}{1.5\times 4 + \frac{4}{3}\times(1.5)^2} = \frac{0.215}{9}$$

$$\therefore \frac{\delta V}{V}\times 100 = \frac{0.215}{9}\times 100 = 2.389\%$$

Hence the percentage change in the volume of the balloon is 2.389.

Q-11 In estimating the cost of a pile of bricks measured as 6m × 50m × 4m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m^3 and bricks cost ₹100 per 1000, find the approximate error in the cost. [UKTU-2010].

Sol^n Let x, y and z m be the length, breadth and height of the pile so that its Volume $V = xyz$.

Taking log, we get $\log V = \log x + \log y + \log z$

On diff.,
$$\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$\Rightarrow \frac{\delta V}{V}\times 100 = \frac{\delta x}{x}\times 100 + \frac{\delta y}{y}\times 100 + \frac{\delta z}{z}\times 100 = 1+1+1 = 3.$$

$$\Rightarrow \frac{\delta V}{V}\times 100 = 3$$

$$\Rightarrow \delta V = \frac{3}{100}\times V = \frac{3}{100}\times(1200) = 36\,m^3. \quad [\text{Since } V = 6\times 50\times 4 = 1200\,m^3]$$

Since the count is 12 bricks in 1 m^3 then the number of bricks in $\delta V = 36\times 12 = 432$.

Thus error in cost $= 432\times\frac{100}{1000} = ₹\,43.20$.

Q-12 In estimating the number of bricks in a pile which is measured to be (5m×10m×5m), the count of bricks is taken as 100 bricks per m^3. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is ₹ 2000 per thousand bricks.

(4)

Q-13→ The work that must be done to propel a ship of displacement D for a distance 'S' in time 't' is proportional to $\frac{S^2 D^{2/3}}{t^2}$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time diminished by 1% and the distance diminished by 2%.

Solⁿ→ Let the work done be W then $W \propto \frac{S^2 D^{2/3}}{t^2}$

$$\Rightarrow W = K \frac{S^2 D^{2/3}}{t^2} \quad \text{where } K \text{ is a constant.}$$

Taking log, $\log W = \log K + 2 \log S + \frac{2}{3} \log D - 2 \log t$.

On diff. $\frac{\delta W}{W} = 2 \frac{\delta S}{S} + \frac{2}{3} \frac{\delta D}{D} - 2 \frac{\delta t}{t}$.

$$\Rightarrow \frac{\delta W}{W} \times 100 = 2 \frac{\delta S}{S} \times 100 + \frac{2}{3} \frac{\delta D}{D} \times 100 - 2 \frac{\delta t}{t} \times 100$$

$$= 2(-2) + \frac{2}{3}(1) - 2(-1) = -\frac{4}{3}.$$ [for diminished −ve sign].

∴ Approximate increase of work $= -\frac{4}{3}\%$.

Q-14→ If $pv^2 = K$ and the relative error in p and v are respectively 0.05 and 0.025, show that the error in K is 10%. (AKTU-2015).

Solⁿ→ Given $pv^2 = K$. Take log both side

$$\Rightarrow \log K = \log p + 2 \log v.$$

On diff, $\frac{\delta K}{K} = \frac{\delta p}{p} + 2 \frac{\delta v}{v}$

$$\Rightarrow \frac{\delta K}{K} = 0.05 + 2 \times 0.025 \qquad \left[\because \frac{\delta p}{p} = 0.05 \ \& \ \frac{\delta v}{v} = 0.025\right]$$

$$= 0.05 + 0.05 = 0.10$$

$$\boxed{\therefore \frac{\delta K}{K} \times 100 = 0.10 \times 100 = 10\%}$$

Mathematical Based Question

Q-1→ Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, if r_1, r_2, r_3 are each in error by +1.2%. [UPTU-2011].

Solⁿ→ We have $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \rightarrow (1)$

On Differentiating, we get

$$-\frac{1}{r^2} \delta r = -\frac{1}{r_1^2} \delta r_1 - \frac{1}{r_2^2} \delta r_2 - \frac{1}{r_3^2} \delta r_3$$

$$\Rightarrow \frac{1}{r}\left(\frac{\delta r}{r} \times 100\right) = \frac{1}{r_1}\left(\frac{\delta r_1}{r_1} \times 100\right) + \frac{1}{r_2}\left(\frac{\delta r_2}{r_2} \times 100\right) + \frac{1}{r_3}\left(\frac{\delta r_3}{r_3} \times 100\right)$$

$$= \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2)$$

$$= 1.2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= 1.2 \times \frac{1}{r} \quad \text{(from (1))}$$

$$\Rightarrow \frac{\delta r}{r} \times 100 = 1.2.$$

Hence % error in r = 1.2.

Q-2→ Find the possible percentage error in computing the parallel resistance r of two resistances r_1 & r_2 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ where the error in both r_1 and r_2 is +2% each. [Ans→ 2%]

Method-II → **Step-1:** Consider the function $u=f(x)$ for single variable & $u=f(x,y)$ for double variable and $u=f(x,y,z)$ for three variable.

Step-2: Round off each term to its nearest integer & find x, dx, y, dy, z, dz etc.

Step-3: Use these values in the formula,

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \quad \text{or} \quad df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz.$$

Step-4: Approximate value of given expression is $\boxed{f+df}$.

Q-1: Compute an approximate value of $(1.04)^{3.01}$. [UPTU-2014].

Sol^n: Let $f(x,y)=x^y \rightarrow ①$

As $1.04=1+.04$

$3.01=3+.01$ then let $x=1$ & $dx=.04$

$y=3$ & $dy=.01$

Now $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy.$

$\Rightarrow df=yx^{y-1}dx+x^y\log x\, dy.$

$=3(1)^{3-1}(.04)+1^3(\log 1)(0.01)=.12.$

& $f(1,3)=1^3=1.$

Then Approximate value of $(1.04)^{3.01}=f+df$

$=1+.12$

$=1.12.$

Q-2: Find the approximate value of x^y, where $x=4.01$ and $y=2.99$.

Q-3: Find the approximate value of $(2.4)^{3.1}$.

Q-4: Find approximate value of $[(0.98)^2+(2.01)^2+(1.94)^2]^{1/2}$ [UPTU-2015].

Sol^n: Let $f(x,y,z)=(x^2+y^2+z^2)^{1/2}$.

Taking $x=1$, $dx=-.02$

$y=2$, $dy=.01$

$z=2$, $dz=-.06$

Now $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=(x^2+y^2+z^2)^{-1/2}x\,dx+(x^2+y^2+z^2)^{-1/2}y\,dy+(x^2+y^2+z^2)^{-1/2}z\,dz$

$\Rightarrow df=(x^2+y^2+z^2)^{-1/2}[x\,dx+y\,dy+z\,dz]$

$=(1^2+2^2+2^2)^{-1/2}[1\times-.02+2\times.01+2\times-.06]$

$=\frac{1}{3}(-.02+.02-.12]=-.04.$

$\therefore$ Approximate value of $[(0.98)^2+(2.01)^2+(1.94)^2]=f(1,2,2)+df$

$=3+(-.04)$

$=2.96.$

Q-5: Compute an approximate value of $[(3.82)^2+2(2.1)^3]^{1/5}$. (UPTU-2014)

(Let $f(x,y)=(x^2+2y^3)^{1/5}$). [Ans → 2.012.]

Q-6: Find the approximate value of $[(3.98)^2+(2.01)^2+3(1.94)^2]^{1/5}$. (UPTU-2014)

Let $f(x,y)=(x^2+y^2+3z^2)^{1/5}$.

Q-7: If $f = x^2y^2z^{1/10}$, find the approximate value of f where $x = 1.99$, $y = 3.01$, $z = 0.92$. (6)

Sol: Let $f(x,y,z) = x^2y^2z^{1/10} \longrightarrow (1)$

Taking $x = 2$ then $dx = -.01$

$y = 3$ then $dy = .01$

$z = 1$ then $dz = -.08$.

Now $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$\Rightarrow df = 2xy^2z^{1/10}dx + 2x^2y\,z^{1/10}dy + \frac{1}{10}x^2y^2z^{-9/10}dz$.

$\Rightarrow df = 2\times2\times3^2\times1^{1/10}\times-.01 + 2\times2^2\times3\times1^{1/10}\times.01 + \frac{1}{10}\times2^2\times3^2\times1^{-9/10}\times-.08$

$= 36\times-.01 + 24\times.01 + 3.6\times-.08$

$= -0.36 + 0.24 - 0.288 = -.408$

$\therefore$ Approximate value of f = $f(2,3,1) + df$

$= 36 - .408 = 35.592$.

Q-8: If $f(x,y) = x^2y^{1/10}$, compute the value of f when $x = 1.99$ & $y = 3.01$.

Q-9: What error in the common logarithm of a number will be produced by an error of 1% in the number? [AKTU-2018]

Sol: Consider x as any number & $y = \log_{10}x$ Then $\delta y = \frac{1}{x}\log_{10}e \times \delta x$

$\therefore \delta y = \left(\frac{\delta x}{x}\times100\right)\left(\frac{1}{100}\log_{10}e\right)$

$= \frac{1}{100}\log_{10}e$ $\left[\because \frac{\delta x}{x}\times100 = 1.\right]$

$\Rightarrow \delta y = \frac{.43429}{100} = .0043429$. [Note $\frac{d}{dx}\log_{10}x = \frac{1}{x}\log_{10}e$]

Q: If the kinetic energy T is given by $T = \frac{1}{2}mv^2$, find approximate the change in T as the mass m changes from 49 to 49.5 & the velocity v changes from 1600 to 1590.

Sol: $T = \frac{1}{2}mv^2$

$\therefore \delta T = \frac{1}{2}[\delta m\, v^2 + m\,2v\,\delta v] \rightarrow (1)$

It is given that m changes from 49 to 49.5.

$\therefore \delta m = 0.5$

Also, v changes from 1600 to 1590, $\therefore \delta v = -10$.

from (1) $\delta T = \frac{1}{2}(1600)^2(0.5) + 49(1600)(-10) = -144000$

Thus T decreases by 144000 units.