MATHEMATICS-I
KAS-103T
Lecture No- 16

* Modulo-II *
Diff. Calculus
[redacted]

Successive differentiation→

Let $y = f(x)$ is a function of x, then its derivative $\frac{dy}{dx} = f'(x)$ can be further differentiate, is said to be successive differentiation.

Let $y = f(x)$

then $\frac{dy}{dx} = f'(x) = Dy = y_1 = y'$

$$\frac{d^2y}{dx^2} = f''(x) = D^2y = y_2 = y''$$

$$\frac{d^3y}{dx^3} = f'''(x) = D^3y = y_3 = y'''$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = f^n(x) = D^ny = y_n = y^{(n)}.$$

where $D = \frac{d}{dx}$

Formulae→

① Find the nth derivative of e^{ax}.

Solⁿ→ Let $y = e^{ax}$

$$y_1 = \frac{dy}{dx} = ae^{ax}$$

$$y_2 = \frac{d^2y}{dx^2} = a^2e^{ax}$$

$$y_3 = \frac{d^3y}{dx^3} = a^3e^{ax}$$

$$\vdots$$

$$y_n = \frac{d^ny}{dx^n} = a^ne^{ax}$$

Hence $\boxed{D^ny = D^ne^{ax} = a^ne^{ax}}$

② Find D^na^x.

Solⁿ→ Let $y = a^x$

then $y_1 = \frac{dy}{dx} = \log a\, a^x$

$$y_2 = \frac{d^2y}{dx^2} = (\log a)^2 a^x$$

$$y_3 = \frac{d^3y}{dx^3} = (\log a)^3 a^x$$

$$\vdots$$

$$y_n = \frac{d^ny}{dx^n} = (\log a)^n a^x.$$

Hence $\boxed{D^ny = D^na^x = (\log a)^n a^x}$

③ Find $D^n x^n$.

Let $y = x^n$

then $y_1 = \frac{dy}{dx} = nx^{n-1}$

$$y_2 = \frac{d^2y}{dx^2} = n(n-1)x^{n-2}$$

$$y_3 = \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$$

$$\vdots$$

$$y_n = \frac{d^ny}{dx^n} = n(n-1)(n-2)\ldots[n-(n-1)]x^{n-n}$$

$$= n(n-1)(n-2)\ldots 2\cdot 1\, x^0$$

$$= \lfloor n$$

Hence $\boxed{D^nx^n = \lfloor n}$

④ Find nth derivative of $(ax+b)^m$.

Sol^n: Let $y = (ax+b)^m$

$$y_1 = \frac{dy}{dx} = a\, m\, (ax+b)^{m-1}$$

$$y_2 = \frac{d^2y}{dx^2} = a^2\, m\,(m-1)\,(ax+b)^{m-2}$$

$$y_3 = \frac{d^3y}{dx^3} = a^3\, m\,(m-1)(m-2)\,(ax+b)^{m-3}$$

$$\vdots$$

$$\boxed{D^n (ax+b)^m = y_n = \frac{d^ny}{dx^n} = a^n\, m(m-1)(m-2)\cdots[m-(n-1)]\,(ax+b)^{m-n}}$$

⑤ Find the nth differential coefficient of $\frac{1}{ax+b}$.

V.a

Sol^n: Let $y = \frac{1}{ax+b}$

then
$$y_1 = \frac{dy}{dx} = \frac{(-1)\,a}{(ax+b)^2} = \frac{(-1)^1 \lfloor 1\; a^1}{(ax+b)^2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{(-1)(-2)\,a^2}{(ax+b)^3} = \frac{(-1)^2 \lfloor 2\; a^2}{(ax+b)^3}$$

$$y_3 = \frac{d^3y}{dx^3} = \frac{(-1)(-2)(-3)\,a^3}{(ax+b)^4} = \frac{(-1)^3 \lfloor 3\; a^3}{(ax+b)^4}$$

$$\vdots$$

$$\boxed{D^n \frac{1}{ax+b} = y_n = \frac{d^ny}{dx^n} = \frac{(-1)^n \lfloor n\; a^n}{(ax+b)^{n+1}}}$$ V.a

⑥ Find the nth differential coefficient of $\log(ax+b)$.

Sol^n: Let $y = \log(ax+b)$

then
$$y_1 = \frac{dy}{dx} = \frac{1}{(ax+b)}\cdot a$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{(-1)\;a^2}{(ax+b)^2} = \frac{(-1)^1\; a^2}{(ax+b)^2}$$

$$y_3 = \frac{d^3y}{dx^3} = \frac{(-1)(-2)\,a^3}{(ax+b)^3} = \frac{(-1)^2 \lfloor 2\; a^3}{(ax+b)^3}$$

$$y_4 = \frac{d^4y}{dx^4} = \frac{(-1)(-2)(-3)\,a^4}{(ax+b)^4} = \frac{(-1)^3 \lfloor 3\; a^4}{(ax+b)^4}$$

$$\vdots$$

$$\boxed{D^n \log(ax+b) = y_n = \frac{(-1)^{n-1} \lfloor n-1\; a^n}{(ax+b)^n}}$$

V.Imp

(7) find $D^n \sin(ax+b)$

Sol^n Let $y = \sin(ax+b)$

then $y_1 = \frac{dy}{dx} = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$y_2 = \frac{d^2y}{dx^2} = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$.

$y_3 = \frac{d^3y}{dx^3} = a^3\cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$

$\vdots$

$y_n = \frac{d^ny}{dx^n} = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$

Hence $\boxed{D^n y = D^n\sin(ax+b) = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)}$

(8) Find $D^n\cos(ax+b)$. (Prove yourself).

$\boxed{D^n\cos(ax+b) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)}$

(9) Find the nth differential coefficient of $e^{ax}\sin(bx+c)$.

Sol^n Let $y = e^{ax}\sin(bx+c)$

then $y_1 = a e^{ax}\sin(bx+c) + b e^{ax}\cos(bx+c) \longrightarrow (1)$

Put $a = r\cos\theta$
$b = r\sin\theta$ then $r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1}\frac{b}{a}$ $\Big\} \longrightarrow (2)$

Using eq^n (2) in (1), we get

$y_1 = e^{ax}[r\cos\theta . \sin(bx+c) + r\sin\theta\cos(bx+c)]$

$\Rightarrow y_1 = r e^{ax}\sin(bx+c+\theta)$

similarly,

$y_2 = r^2 e^{ax}\sin(bx+c+2\theta)$

$y_3 = r^3 e^{ax}\sin(bx+c+3\theta)$

$\vdots$

$y_n = r^n e^{ax}\sin(bx+c+n\theta)$.

Hence $\boxed{y_n = D^n y = D^n e^{ax}\sin(bx+c) = (a^2+b^2)^{n/2} e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)}$

from eq^n (2).

(10) find the nth diff. coefficient of $e^{ax}\cos(bx+c)$.

Try yourself.

Ans $\boxed{D^n e^{ax}\cos(bx+c) = (a^2+b^2)^{n/2} e^{ax}\cos(bx+c+n\tan^{-1}\frac{b}{a})}$

MATHEMATICS-I
KAS-103T
Lecture No – 17

* Module – II *
Diff. Calculus
[redacted].

Successive diff. Examples →

Type-1 → Use of Partial fraction

Note → ① If $f(x) = \frac{p(x)}{q(x)}$ & $\deg p(x) < \deg q(x)$. Then partial fraction used to siplify.

② If $\deg p(x) = \deg q(x)$ or $\deg p(x) > \deg q(x)$ then first divide and then use partial fraction.

Ex → ① find the nth derivative of $\frac{2x+1}{(2x-1)(2x+3)}$. (AKTU-2015).

Solⁿ → Let $y = \frac{2x+1}{(2x-1)(2x+3)}$

$$= \frac{1/2}{2x-1} + \frac{1/2}{2x+3} \quad \text{(By Partial fraction)}$$

$$\Rightarrow y = \frac{1}{2}\left[\frac{1}{2x-1} + \frac{1}{2x+3}\right] \longrightarrow ①$$

Differentiate eqⁿ ① up to nth time, we get

$$D^n y = \frac{1}{2}\left[D^n \frac{1}{2x-1} + D^n \frac{1}{2x+3}\right]$$

$$\Rightarrow y_n = \frac{1}{2}\left[\frac{(-1)^n \lfloor n \; 2^n}{(2x-1)^{n+1}} + \frac{(-1)^n \lfloor n \; 2^n}{(2x+3)^{n+1}}\right]. \qquad \left[\because D^n \frac{1}{ax+b} = \frac{(-1)^n \lfloor n \; a^n}{(ax+b)^{n+1}}\right]$$

Ex-2 → find nth derivative of $\frac{x^2}{(x+2)(2x+3)}$. (AKTU-2011)

Solⁿ → Let $y = \frac{x^2}{(x+2)(2x+3)}$ [Here deg(Nr.) = deg(Dr.). So divide first].

$$\Rightarrow y = \frac{1}{2} - \frac{\left(\frac{7}{2}x+3\right)}{(x+2)(2x+3)} \quad \text{(by divide)}.$$

$$\Rightarrow y = \frac{1}{2} - \left[\frac{4}{x+2} - \frac{9/2}{2x+3}\right]$$

$$\Rightarrow y = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2}\,\frac{1}{2x+3} \longrightarrow ①$$

$$\therefore D^n y = 0 - 4\, D^n \frac{1}{x+2} + \frac{9}{2}\, D^n \frac{1}{2x+3} \qquad \left[\because D^n \frac{1}{ax+b} = \frac{(-1)^n \lfloor n \; a^n}{(ax+b)^{n+1}}\right]$$

$$\Rightarrow y_n = -4\, \frac{(-1)^n \lfloor n}{(x+2)^{n+1}} + \frac{9}{2}\, \frac{(-1)^n \lfloor n \; 2^n}{(2x+3)^{n+1}}$$

Ex-54 Find the nth order derivative of $\tan^{-1}\frac{x}{a}$. Prove that
$y_n = (-1)^{n-1} \lfloor n-1 \; a^{-n} \sin^n\theta \sin n\theta$ where $\theta = \tan^{-1}\frac{a}{x}$. (AKTU-2009)

Soln Let $y = \tan^{-1}\frac{x}{a}$

then $y_1 = \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a} = \frac{a}{x^2+a^2} = \frac{a}{(x-ia)(x+ia)}$

$$\Rightarrow y_1 = \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right] \quad \text{(by Partial fraction)} \longrightarrow (1)$$

Diff. eqn (1) both side upto $\boxed{(n-1)}$ time, we get

$$\therefore D^{n-1}y_1 = \frac{1}{2i}\left[D^{n-1}\frac{1}{x-ia} - D^{n-1}\frac{1}{x+ia}\right]$$

$$\Rightarrow y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}\lfloor n-1}{(x-ia)^n} - \frac{(-1)^{n-1}\lfloor n-1}{(x+ia)^n}\right] \quad \left[\because D^n\frac{1}{ax+b} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{(-1)^{n-1}\lfloor n-1}{2i}\left[(x-ia)^{-n} - (x+ia)^{-n}\right] \longrightarrow (2)$$

Put $x = r\cos\theta$, $a = r\sin\theta$

$\therefore a^2+x^2 = r^2$ & $\theta = \tan^{-1}\frac{a}{x}$ $\Big\} \longrightarrow (3)$

Using (3) in (2), we get

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{2i}\left[(r\cos\theta - ir\sin\theta)^{-n} - (r\cos\theta + ir\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}\lfloor n-1}{2i} r^{-n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}\lfloor n-1}{2i} r^{-n}\left[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)\right]$$

[By D. Movier Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$]

$$\Rightarrow y_n = \frac{(-1)^{n-1}\lfloor n-1}{2i} r^{-n} \cdot 2i\sin n\theta$$

$$\Rightarrow y_n = \frac{(-1)^{n-1}\lfloor n-1}{2i} \cdot \left(\frac{a}{\sin\theta}\right)^{-n} 2i\sin n\theta \quad [\because a = r\sin\theta]$$

$$\Rightarrow \boxed{y_n = (-1)^{n-1}\lfloor n-1 \; a^{-n}\sin^n\theta \sin n\theta \quad \text{where } \theta = \tan^{-1}\frac{a}{x}}$$

Ex-4: find the nth derivative of

(i) $\tan^{-1} x$ (Put $a=1$ in last question) (ii) $\tan^{-1}\frac{2x}{1-x^2}$ $\boxed{\text{Hint } \tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}x}$

(iii) $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ (iv) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

Hint-(iii) → Let $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ Put $x=\tan\theta$ or $\theta = \tan^{-1}x$.

$$= \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left[\frac{1-1+2\sin^2\theta/2}{2\sin\frac{\theta}{2}\cos\theta/2}\right]$$

$$= \tan^{-1}\tan\frac{\theta}{2}$$

$$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x.$$

$\Rightarrow y = \frac{1}{2}\tan^{-1}x$ (Solve yourself Now).

Hint(iv) → Let $y = \tan^{-1}\frac{1+x}{1-x}$ Put $x=\tan\theta$, $\theta = \tan^{-1}x$.

$$\Rightarrow y = \tan^{-1}\frac{1+\tan\theta}{1-\tan\theta}$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right] \quad \left[\text{as } \tan\frac{\pi}{4}=1\right]$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\tan^{-1}x.$$

$\Rightarrow y = \frac{\pi}{4}+\tan^{-1}x$ (Solve yourself Now).

Ex-5: Find the nth derivative of the following functions.

(i) $\frac{1}{x^2-a^2}$ (ii) $\frac{1}{x^2+a^2}$ (AKTU-2011) (iii) $\frac{5x+12}{x^2+5x+6}$

(iv) If $y = \frac{1}{1-5x+6x^2}$, find y_n.

(v) If $y = \frac{x^2}{(x-1)^2(x+2)}$, find $D^n y$. (Hint Put $x-1=Z$).

Solⁿ (v) Put $x-1 = z$ or $x = z+1$.

$$y = \frac{x^2}{(x-1)^2(x+2)} = \frac{(z+1)^2}{z^2(z+3)}$$

$$\Rightarrow y = \frac{1}{z^2}\left[\frac{z^2+2z+1}{z+3}\right] = \frac{1}{z^2}\left[\frac{1+2z+z^2}{3+z}\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{3} + \frac{5}{9}z + \frac{4}{9}\frac{z^2}{3+z}\right] \text{ (By divide)}$$

$$= \frac{1}{3z^2} + \frac{5}{9z} + \frac{4}{9(z+3)}$$

$$\Rightarrow y = \frac{1}{3}\frac{1}{(x-1)^2} + \frac{5}{9(x-1)} + \frac{4}{9(x+2)}$$

Hence $D^n y = \frac{1}{3} D^n \frac{1}{(x-1)^2} + \frac{5}{9} D^n \frac{1}{(x-1)} + \frac{4}{9} D^n \frac{1}{x+2}$

$$\Rightarrow y_n = \frac{1}{3}\frac{(-1)^n \lfloor n+1}{(x-1)^{n+2}} + \frac{5}{9}\frac{(-1)^n \lfloor n}{(x-1)^{n+1}} + \frac{4}{9}\frac{(-1)^n \lfloor n}{(x+2)^{n+1}}$$

Ex- Prove that the value of the nth diff. coefficient of $\frac{x^3}{x^2-1}$ for $x=0$ is 0 if n is even & is $-\lfloor n$ if n is odd and greater than 1.

Ex- find the nth derivative of $\frac{ax+b}{x+a}$. (AKTU-2008).

Type-II →

Ex-1- If $y = \sin 2x \sin 3x$, find y_n.

Solⁿ- Given $y = \sin 2x \sin 3x$

$$= \frac{1}{2}(2 \sin 3x \sin 2x)$$

$$\Rightarrow y = \frac{1}{2}[\cos x - \cos 5x] \longrightarrow (1) \quad [\because 2\sin A \sin B = \cos(A-B) - \cos(A+B).$$

then $D^n y = \frac{1}{2}[D^n \cos x - D^n \cos 5x]$

$$\Rightarrow y_n = \frac{1}{2}\left[\cos\left(x + \frac{n\pi}{2}\right) - 5^n \cos\left(5x + \frac{n\pi}{2}\right)\right]$$

$$[\because D^n \cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

Ex-2- find y_n if

(i) $y = \sin^2 x$ (ii) $\cos^2 x$ (iii) $\sin^3 x$ (iv) $\cos^3 x$. (AKTU-2004)

(iv) $\sin x \cdot \cos 2x$ (v) $\cos x \cdot \cos 2x$ (vi) $\sin^3 x \cos^2 x$.

Ex 3 find n^{th} derivative of $\sin^2 x \cos^3 x$.

Sol^n: Let $y = \sin^2 x \cos^3 x$

$$= (\sin x \cos x)^2 \cdot \cos x$$

$$= \left(\frac{1}{2} 2\sin x \cos x\right)^2 \cdot \cos x$$

$$= \frac{1}{4} \sin^2 2x \cos x$$

$$= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)\cos x \qquad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$= \frac{1}{8}[\cos x - \cos 4x \cos x]$$

$$= \frac{1}{8}\times\frac{1}{2}[2\cos x - 2\cos 4x \cos x]$$

$$\Rightarrow y = \frac{1}{16}[2\cos x - \cos 5x - \cos 3x] \longrightarrow (1)$$

$$[\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

Diff. eq^n (1) nth time

$$D^n y = \frac{1}{16}[2D^n \cos x - D^n \cos 5x - D^n \cos 3x]$$

$$\Rightarrow y_n = \frac{1}{16}\left[2\cos\left(x+\frac{n\pi}{2}\right) - 5^n \cos\left(5x+\frac{n\pi}{2}\right) - 3^n \cos\left(3x+\frac{n\pi}{2}\right)\right]$$

$$\left[\because D^n \cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)\right]$$

Ex-4: If $y = \cos mx + \sin mx$ then show that $y_n = m^n[1+(-1)^n \sin 2mx]^{1/2}$

V.Imp Hence show that $y_8(\pi) = \left(\frac{1}{2}\right)^{31/2}$ when $m = \frac{1}{4}$. (AKTU-2012, 2015, 2018).

Sol^n: Given $y = \sin mx + \cos mx \longrightarrow (1)$

Diff. eq^n (1) upto nth time, we get

$$D^n y = D^n \sin mx + D^n \cos mx$$

$$\Rightarrow y_n = m^n \sin\left(mx+\frac{n\pi}{2}\right) + m^n \cos\left(mx+\frac{n\pi}{2}\right)$$

$$\left[\because D^n \cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right) \text{ \& } D^n \sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow y_n = m^n\left[\left\{\sin\left(mx+\frac{n\pi}{2}\right) + \cos\left(mx+\frac{n\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$= m^n\left[\sin^2\left(mx+\frac{n\pi}{2}\right) + \cos^2\left(mx+\frac{n\pi}{2}\right) + 2\sin\left(mx+\frac{n\pi}{2}\right)\cos\left(mx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$\Rightarrow y_n = m^n\left[1 + \sin 2\left(mx+\frac{n\pi}{2}\right)\right]^{1/2}$$

$$= m^n[1 + \sin(2mx + n\pi)]^{1/2}$$

$$\boxed{y_n = m^n[1+(-1)^n \sin 2mx]^{1/2}} \longrightarrow (2)$$

Now Put $n=8,\ x=\pi$ & $m=\frac{1}{4}$ in eqn (2)

$$y_8(\pi) = \left(\frac{1}{4}\right)^8\left[1+(-1)^8 \sin 2\cdot\frac{1}{4}\cdot\pi\right]^{\frac{1}{2}}$$

$$= \frac{1}{4^8}\left[1+1\cdot\sin\frac{\pi}{2}\right]^{1/2}$$

$$= \frac{1}{2^{16}}\sqrt{2} = \frac{1}{2^{16-1/2}} = \frac{1}{2^{31/2}}$$

$$\Rightarrow \boxed{y_8(\pi) = \frac{1}{2^{31/2}}}.$$

Type-III →

Ex-1 → find the nth differential coefficient of $e^x \sin^3 x$.

Sol^n → Let $y = e^x \sin^3 x$

$$= e^x\,\frac{1}{4}\left[3\sin x - \sin 3x\right]$$

$[\because \sin 3A = 3\sin A - 4\sin^3$

$$= \frac{1}{4}\left[e^x\cdot 3\sin x - e^x \sin 3x\right]$$

$$\Rightarrow y = \frac{3}{4}e^x \sin x - \frac{1}{4}e^x \sin 3x \longrightarrow (1)$$

Diff. upto nth time eqn (1), we get

$$D^n y = \frac{3}{4}D^n e^x \sin x - \frac{1}{4}D^n e^x \sin 3x$$

$$\Rightarrow y_n = \frac{3}{4}\left[(1^2+1^2)^{n/2} e^x \sin(x+n\tan^{-1}1)\right]$$

$$-\frac{1}{4}\left[(1^2+3^2)^{n/2} e^x \sin\left(3x+n\tan^{-1}\frac{3}{1}\right)\right]$$

$$\left[\because D^n e^{ax}\sin(bx+c) = (a^2+b^2)^{n/2} e^{ax}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)\right]$$

$$\Rightarrow y_n = \frac{3}{4}\,2^{n/2} e^x \sin\left(x+\frac{n\pi}{4}\right) - \frac{1}{4}\,10^{n/2}\cdot e^x \sin(3x+n\tan^{-1}3)$$

Ex → find y_n if

(i) $y = e^x \cos x$ (ii) $e^x \sin x$ (iii) $e^x \sin x \cos x$

(iv) $e^{3x}\sin^2 2x$ (v) $e^x \cos^3 x$.

[Hint $\cos 3A = 4\cos^3 A - 3\cos A$].

Type-IV :-

Ex-1 :- find the nth differential coefficient of $\log(ax+x^2)$.

Solⁿ :- Let $y = \log(ax+x^2)$

$\Rightarrow y = \log x(a+x)$

$\Rightarrow y = \log x + \log(a+x) \rightarrow (1)$

Diff. nth time,

$D^n y = D^n \log x + D^n \log(x+a)$.

$\Rightarrow y_n = \frac{(-1)^{n-1} \lfloor n-1 \cdot 1^n}{x^n} + \frac{(-1)^{n-1} \lfloor n-1 \; 1^n}{(x+a)^n}$

$\Rightarrow y_n = (-1)^{n-1} \lfloor n-1 \left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right]$. $\left[\because D^n \log(ax+b) = \frac{(-1)^{n-1} \lfloor n-1 \; a^n}{(ax+b)^n}\right]$

Ex-2 :-

If $y = x \log \frac{x-1}{x+1}$, show that

$$y_n = (-1)^{n-2} \lfloor n-2 \left[\frac{(x-n)}{(x-1)^n} - \frac{(x+n)}{(x+1)^n}\right].$$

12th formulae :-

(1) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(2) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(3) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(4) $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

(5) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

(6) $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

(7) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

(8) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

(9) $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$.

(10) $\sin 3A = 3\sin A - 4\sin^3 A$

(11) $\cos 3A = 4\cos^3 A - 3\cos A$.

V.Imp

Leibnitz's Theorem → (nth time differentiation of product of two funcⁿ)

If u and v are two functions of x, having derivatives of the nth order, then

$$\boxed{\begin{aligned} D^n(u \cdot v) &= \frac{d^n}{dx^n}(u \cdot v) \\ &= D^n u \cdot v + {}^nC_1 D^{n-1}u \cdot Dv + {}^nC_2 D^{n-2}u \cdot D^2 v + \cdots + {}^nC_r D^{n-r}u \cdot D^r v \\ &\quad + \cdots + {}^nC_n u \cdot D^n v. \end{aligned}}$$

Ex-1 → Find the nth derivative of $x^2 \sin 5x$ and find y_n at $x=0$. (AKTU-2008)

Solⁿ → Let $y = x^2 \sin 5x. \longrightarrow (1)$

By L.T, we have

$$D^n(u \cdot v) = D^n u \cdot v + {}^nC_1 D^{n-1}u \cdot Dv + {}^nC_2 D^{n-2}u \cdot D^2 v + \cdots + u \cdot D^n v.$$

Take $u = \sin 5x$ & $v = x^2$ and use L.T in (1), we get

$$D^n y = D^n[\sin 5x \cdot x^2]$$

$$\Rightarrow y_n = D^n \sin 5x \cdot x^2 + {}^nC_1 D^{n-1} \sin 5x \cdot Dx^2 + {}^nC_2 D^{n-2} \sin 5x \cdot D^2 x^2$$

$$\Rightarrow y_n = 5^n \sin\left(5x + \frac{n\pi}{2}\right) \cdot x^2 + n\, 5^{n-1} \sin\left(5x + \frac{(n-1)\pi}{2}\right) \cdot 2x$$
$$+ \frac{n(n-1)}{\lfloor 2} 5^{n-2} \sin\left(5x + \frac{(n-2)\pi}{2}\right) \cdot 2 \longrightarrow (2)$$

[As $D^n \sin(ax+b) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$.

Put $x = 0$ in (2), we get

$$\boxed{(y_n)_0 = n(n-1)\, 5^{n-2} \sin\frac{(n-2)\pi}{2}}.$$

Ex-2 → Find y_n at $x=0$ if $y = x^2 \sin x$. (AKTU-2009).

Ex-3 → If $y = x^2 e^x$ then find y_n.

Ex-4 → Find the nth derivative of

(i) $e^x \log x$ (ii) $x^3 e^{ax}$ (iii) $x^3 \cos x$.

Ex-5 → If $y = x^2 \sin x$, prove that

$$\frac{d^n y}{dx^n} = (x^2 - n^2 + n) \sin\left(x + \frac{n\pi}{2}\right) - 2nx \cos\left(x + \frac{n\pi}{2}\right).$$ (AKTU-2014).

Working Rule :

1. Put the given func^n equal to y. (If possible)
2. Find $y_1 = \frac{dy}{dx}$.
3. Then (i) Take LCM (if possible)
   (ii) Square both sides if square roots are there.
   (iii) Try to get y in R.H.S (if possible).
4. Again diff. both sides w.r.t x to get an equation in y, y_1, y_2.
5. Apply Leibnitz theorem.

Ex-1: If $y = \sin\log(x^2+2x+1)$, prove that

$$(1+x)^2 y_{n+2} + (2n+1)(1+x)y_{n+1} + (n^2+4)y_n = 0 .$$ (AKTU-2013) 2009.

Sol^n: Given $y = \sin\log(x^2+2x+1) \longrightarrow (1)$

Then $y_1 = \cos\log(x^2+2x+1)\cdot \dfrac{1}{x^2+2x+1}\cdot(2x+2)$

$\Rightarrow y_1 = \cos\log(x^2+2x+1)\cdot \dfrac{2(x+1)}{(x+1)^2}$

$\Rightarrow y_1(x+1) = 2\cos\log(x^2+2x+1)$

Squaring both side,

$y_1^2(x+1)^2 = 4\cos^2\log(x^2+2x+1) = 4\left[1-\sin^2\log(x^2+2x+1)\right]$

$\Rightarrow y_1^2(x+1)^2 = 4(1-y^2)$ (from (1)) $\longrightarrow (2)$

Diff. eq^n (2) both side w.r.t x, we get

$2y_1\cdot y_2(x+1)^2 + y_1^2\cdot 2(x+1) = 4(-2y\cdot y_1)$

$\Rightarrow y_2\cdot(x+1)^2 + y_1\cdot(x+1) = -4y$.

$\Rightarrow y_2\cdot(x+1)^2 + y_1\cdot(x+1) + 4y = 0 \longrightarrow (3)$

Diff. eq^n (3) n times by Leibnitz Theorem, we get

$D^n[y_2\cdot(x+1)^2] + D^n[y_1\cdot(x+1)] + 4D^ny = 0$.

$\Rightarrow D^ny_2\cdot(x+1)^2 + {}^nC_1 D^{n-1}y_2\cdot D(x+1)^2 + {}^nC_2 D^{n-2}y_2\cdot D^2(x+1)^2$
$\quad + D^ny_1\cdot(x+1) + {}^nC_1 D^{n-1}y_1\cdot D(x+1) + 4D^ny = 0$.

$\Rightarrow y_{n+2}\cdot(x+1)^2 + n\,y_{n+1}\cdot 2(x+1) + \dfrac{n(n-1)}{\lfloor 2}\,y_n\cdot 2$
$\quad + y_{n+1}\cdot(x+1) + n\,y_n\cdot 1 + 4y_n = 0$

$\Rightarrow (x+1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + [n^2-n+n+4]\, y_n = 0.$

$\Rightarrow (x+1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (n^2+4)\, y_n = 0.$

Hence proved.

Ex-21 If $y^{1/m} + y^{-1/m} = 2x$, prove that

$(x^2-1)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2-m^2)\, y_n = 0$ (AKTU-2011, 2015, 2020).

Sol^n Given $y^{1/m} + y^{-1/m} = 2x$ Let $y^{1/m} = t$

$\Rightarrow t + t^{-1} = 2x$

$\Rightarrow t + \frac{1}{t} = 2x \Rightarrow t^2+1 = 2xt$

$\Rightarrow t^2 - 2xt + 1 = 0$

$\therefore t = \frac{2x \pm \sqrt{4x^2 - 4\cdot1\cdot1}}{2\cdot 1}$

$\Rightarrow t = x \pm \sqrt{x^2-1}$

$\Rightarrow y^{1/m} = x \pm \sqrt{x^2-1}$

$\Rightarrow y = [x \pm \sqrt{x^2-1}]^m \longrightarrow (1)$

If $y = [x + \sqrt{x^2-1}]^m$

then $y_1 = m[x+\sqrt{x^2-1}]^{m-1} \cdot \left[1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right]$

$= m[x+\sqrt{x^2-1}]^{m-1} \frac{[\sqrt{x^2-1} + x]}{\sqrt{x^2-1}}$

$= \frac{m[x+\sqrt{x^2-1}]^m}{\sqrt{x^2-1}}$

$\Rightarrow y_1 = \frac{my}{\sqrt{x^2-1}}$. And similarly, if $y = [x - \sqrt{x^2-1}]^m$

then $y_1 = \frac{-my}{\sqrt{x^2-1}}$

In both case squaring both side,

$y_1^2 = \frac{m^2y^2}{(x^2-1)} \Rightarrow y_1^2(x^2-1) = m^2y^2 \longrightarrow (2)$

$\Rightarrow$ ~~y_1~~ Diff. eq^n (2) again, we get

$2y_1 \cdot y_2 (x^2-1) + y_1^2 \cdot 2x = m^2 \cdot 2yy_1$

$\Rightarrow y_2(x^2-1) + y_1 \cdot x - m^2 y = 0 \longrightarrow (3)$

Diff. eqⁿ ③ nth time by Leibnitz theorem, we get

$$D^n[y_2\cdot(x^2-1)] + D^n[y_1\cdot x] - m^2 D^n y = 0$$

$$\Rightarrow D^n y_2\cdot(x^2-1) + {}^nC_1 D^{n-1}y_2\cdot D(x^2-1) + {}^nC_2 D^{n-2}y_2\cdot D^2(x^2-1)$$
$$+ D^n y_1\cdot x + {}^nC_1 D^{n-1}y_1\cdot Dx - m^2 D^n y = 0$$

$$\Rightarrow y_{n+2}(x^2-1) + n\, y_{n+1}\cdot 2x + \frac{n(n-1)}{\lfloor 2}\, y_n\cdot 2 + y_{n+1}\cdot x$$
$$+ n\, y_n\cdot 1 - m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(x^2-1) + (2n+1)x\,y_{n+1} + (n^2-n+n-m^2)y_n = 0.$$

$$\Rightarrow y_{n+2}(x^2-1) + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0.$$ Hence proved.

Ex-3→ If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m$, prove that

$$x^2 y_{n+2} + (2n+1)x\,y_{n+1} + (n^2+m^2)y_n = 0.$$ (AKTU-2003).

Solⁿ→ Given $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{m}\right)^m$

$$\Rightarrow \cos^{-1}\left(\frac{y}{b}\right) = m\log x - m\log m \longrightarrow ①$$

On diff.,
$$\frac{-1}{\sqrt{1-\frac{y^2}{b^2}}}\cdot\frac{y_1}{b} = \frac{m}{x}$$

$$\Rightarrow \frac{-y_1}{\sqrt{b^2-y^2}} = \frac{m}{x} \Rightarrow -y_1\cdot x = m\sqrt{b^2-y^2}$$

On squaring both side,

$$y_1^2 x^2 = m^2(b^2-y^2) \longrightarrow ②$$

Again diff. eqⁿ ②,

$$2y_1\cdot y_2\cdot x^2 + y_1^2\cdot 2x = m^2(-2y\,y_1).$$

$$\Rightarrow y_2\cdot x^2 + y_1\cdot x + m^2 y = 0 \longrightarrow ③$$

Diff. eqⁿ ③ n time by L.T, we get

$$D^n[y_2\cdot x^2] + D^n[y_1\cdot x] + m^2 D^n y = 0$$

$$\Rightarrow D^n y_2\cdot x^2 + {}^nC_1 D^{n-1}y_2\cdot Dx^2 + {}^nC_2 D^{n-2}y_2\cdot D^2x^2$$
$$+ D^n y_1\cdot x + {}^nC_1 D^{n-1}y_1\cdot Dx + m^2 D^n y = 0$$

$$\Rightarrow y_{n+2}\cdot x^2 + n\,y_{n+1}\cdot 2x + \frac{n(n-1)}{\lfloor 2}\,y_n\cdot 2x + y_{n+1}\cdot x + n\,y_n\cdot 1$$
$$+ m^2 y_n = 0$$

$$y_{n+2} \cdot x^2 + (2n+1)\, x y_{n+1} + (n^2+m^2)\, y_n = 0$$

Hence proved.

* Home Assignments *

Ex-4: If $y = a\cos(\log x) + b\sin(\log x)$, prove that

i) $x^2 y_2 + x y_1 + y = 0$ ii) $x^2 y_{n+2} + (2n+1)\, x y_{n+1} + (n^2+1)\, y_n = 0$.

(AKTU-2005)

Ex-5: If $y = (x^2-1)^n$, prove that

$$(x^2-1)\, y_{n+2} + 2x\, y_{n+1} - n(n+1)\, y_n = 0$$ (AKTU-2010, 2007).

Ex-6: If $y = (1-x)^{-\alpha} e^{-\alpha x}$, prove that

$$(1-x)\, y_{n+1} - (n+\alpha x)\, y_n - n\alpha\, y_{n-1} = 0$$ (AKTU-2011)

(Hint → find y_1 & then use Leibnitz theorem).

Ex-7: If $y = \left(\frac{1+x}{1-x}\right)^{1/2}$, prove that

$$(1-x^2)\, y_n - [2(n-1)x+1]\, y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0$$

(AKTU-2012).

Ex-8: If $y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$, prove that

$$(a^2+x^2)\, y_{n+2} + 2(n+1)\, x y_{n+1} + n(n+1)\, y_n = 0$$ (AKTU-2012).

Special Examples

Ex-1: If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, then prove that

$$I_n = n I_{n-1} + \lfloor n-1$$ (AKTU-2017).

Sol^n: Given $I_n = \frac{d^n}{dx^n}(x^n \log x) \longrightarrow (1)$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[n x^{n-1}\log x + x^n \cdot \frac{1}{x}\right]$$

$$= n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$\Rightarrow \boxed{I_n = n I_{n-1} + \lfloor n-1}$$

$$\left[(\text{from } (1)) \ \& \ \frac{d^n}{dx^n} x^n = D^n x^n = \lfloor n\right]$$

Ex-2: Find y_n if $y = x^{n-1}\log x$. (AKTU-2010, 2012, 2018).

Sol^n: Given $y = x^{n-1}\log x \longrightarrow (1)$

Diff. eq^n (1) w.r.t x both side we get

$$y_1 = (n-1)x^{n-2}\log x + x^{n-1}\cdot\frac{1}{x}$$

$$\Rightarrow y_1 = (n-1)\frac{x^{n-1}\log x}{x} + \frac{x^{n-1}}{x}$$

$$\Rightarrow y_1\cdot x = (n-1)\,y + x^{n-1} \longrightarrow (2) \quad (\text{from } (1))$$

Differentiating eq^n (2) $(n-1)$ time by Leibnitz theorem,

$$D^{n-1}(y_1\cdot x) = (n-1)D^{n-1}y + D^{n-1}x^{n-1}$$

$$\Rightarrow D^{n-1}y_1\cdot x + {}^{n-1}C_1\, D^{n-2}y_1\cdot Dx = (n-1)D^{n-1}y + \lfloor n-1 \qquad [\because D^n x^n = \lfloor n]$$

$$\Rightarrow y_n\cdot x + (n-1)\,y_{n-1} = (n-1)\,y_{n-1} + \lfloor n-1$$

$$\Rightarrow y_n x = \lfloor n-1 \Rightarrow \boxed{y_n = \frac{\lfloor n-1}{x}}$$

Ex-3: If $y = x^n\log x$, prove that $y_{n+1} = \frac{\lfloor n}{x}$.

MATHEMATICS-I | Module-II
KAS-103T | Diff. Calculus-I
Lecture No. 19 | [redacted]

Determination of nth derivative of a function at $x=0$

Working Rule→

① Put the given function equal to y. (If possible) ⟶ ①

② Find $y_1 = \frac{dy}{dx}$. ⟶ ②

③ Then (i) Take LCM (If possible)
(ii) Square both sides if square roots are there.
(iii) Try to get y in R.H.S (If possible)

④ Again differentiate both sides w.r.t x to get an equation in y, y_1 & y_2. ⟶ ③

⑤ Diff. both side n times w.r.t x by Leibnitz Theorem. ⟶ ④

⑥ Put $x=0$ in equations ①, ②, ③ & ④ and find $(y)_0$, $(y_1)_0$, $(y_2)_0$ and relation in $(y_{n+2})_0$, $(y_{n+1})_0$ & $(y_n)_0$. ⟶ ⑤

⑦ Put $n=1,2,3,4,5...$ in eqn ⑤ and find $(y_3)_0$, $(y_4)_0$, $(y_5)_0$ . . . .

⑧ Using the above equations, discuss the two cases when n is even and when n is odd.

Ex-1→ If $y = \sin(a\sin^{-1}x)$ then prove that
$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-a^2)y_n = 0$. Also find $y_n(0)$.
[AKTU-2011, 2014, 2018]

Soln Given $y = \sin(a\sin^{-1}x)$ ⟶ ①

$\therefore \quad y_1 = \cos(a\sin^{-1}x)\cdot\frac{a}{\sqrt{1-x^2}}$ ⟶ ②

$\Rightarrow y_1\sqrt{1-x^2} = a\cos(a\sin^{-1}x)$

Squarring both side, we get

$y_1^2(1-x^2) = a^2\cos^2(a\sin^{-1}x) = a^2[1-\sin^2(a\sin^{-1}x)]$

$\Rightarrow y_1^2(1-x^2) = a^2(1-y^2)$ (from ①)

Diff. above eqn again, we get

$2y_1\cdot y_2(1-x^2) - y_1^2\cdot 2x = a^2(-2yy_1)$.

$\Rightarrow \quad y_2(1-x^2) - y_1 x + a^2 y = 0 \longrightarrow (3)$

Differentiate eq^n (3) nth time by Leibnitz Theorem, we get

$$D^n[y_2\cdot(1-x^2)] - D^n[y_1\cdot x] + a^2 D^n y = 0.$$

$$\Rightarrow D^n y_2\cdot(1-x^2) + {}^nC_1\, D^{n-1}y_2\cdot D(1-x^2) + {}^nC_2\, D^{n-2}y_2\cdot D^2(1-x^2)$$
$$- D^n y_1\cdot x - {}^nC_1\, D^{n-1}y_1\cdot Dx + a^2 D^n y = 0.$$

$$\Rightarrow y_{n+2}(1-x^2) + n\, y_{n+1}\cdot(-2x) + \frac{n(n-1)}{\lfloor 2}\, y_n\cdot(-2)$$
$$- y_{n+1}\cdot x - n\, y_n\cdot 1 + a^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - (2n+1)x\,y_{n+1} + [-n^2+n-n+a^2]\,y_n = 0.$$

$$\Rightarrow y_{n+2}(1-x^2) - (2n+1)x\,y_{n+1} - (n^2-a^2)\,y_n = 0 \longrightarrow (4).$$

Now Put $x=0$ in (1), (2), (3) & (4), we get

$$y(0) = 0$$
$$y_1(0) = a$$
$$y_2(0) = -a^2 y(0) = 0.$$
$$y_{n+2}(0) = (n^2-a^2)\,y_n(0) \longrightarrow (5)$$

Put $n=1,2,3,4,\ldots$ in eq^n (5), we get

$$\cancel{y_3(0)} \quad y_3(0) = (1^2-a^2)\,y_1(0) = (1^2-a^2)\cdot a$$
$$y_4(0) = (2^2-a^2)\,y_2(0) = 0$$
$$y_5(0) = (3^2-a^2)\,y_3(0) = (3^2-a^2)(1^2-a^2)\,a.$$
$$y_5(0) = (3^2-a^2)\,y_3(0) = 0$$
$$y_6(0) = (4^2-a^2)\,y_4(0) = 0$$
$$\vdots$$

$$\boxed{y_n(0) = \begin{cases} 0 & \text{when } n \text{ is even} \\ [(n-2)^2-a^2]\cdot[(n-4)^2-a^2]\cdots(3^2-a^2)(1^2-a^2)\,a & \text{when } n \text{ is odd.} \end{cases}}$$

Ex-2: If $y = \cos(m\sin^{-1}x)$, prove that

$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2-m^2)\,y_n = 0$. find $(y_n)_0$.

(AKTU-2008)

Ex-3: If $y = \sin^{-1}x$, prove that $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$

Also find the value of y_n when $x=0$. (AKTU- 2013).

V.Imp **Ex-4:** If $y = (\sin^{-1}x)^2$, prove that (AKTU - 2014, 2018, 2016)

(i) $(1-x^2)y_2 - x y_1 - 2 = 0$ (ii) $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$

Also, find the value of nth derivative of y for $x=0$.

Sol^n Given $y = (\sin^{-1}x)^2 \longrightarrow ①$

$\therefore \quad y_1 = 2(\sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}} \longrightarrow ②$

$\Rightarrow y_1\sqrt{1-x^2} = 2\sin^{-1}x$

Squaring both side,

$y_1^2(1-x^2) = 4(\sin^{-1}x)^2 = 4y$ (from ①)

Diff. again,

$2y_1 \cdot y_2(1-x^2) + y_1^2(-2x) = 4y_1$

$\Rightarrow y_2(1-x^2) - y_1 x - 2 = 0 \longrightarrow ③$

Diff. eq^n ③ n th time by Leibnitz Theorem,

$D^n[y_2 \cdot (1-x^2)] - D^n y_1 x - 2D^n \cdot 1 = 0.$

$\Rightarrow D^n y_2 \cdot (1-x^2) + {}^nC_1 D^{n-1}y_2 \cdot D(1-x^2) + {}^nC_2 D^{n-2}y_2 \cdot D^2(1-x^2)$

$- D^n y_1 \cdot x - {}^nC_1 D^{n-1}y_1 \cdot Dx = 0.$

$\Rightarrow y_{n+2}(1-x^2) + n y_{n+1}(-2x) + \frac{n(n-1)}{\lfloor 2} y_n(-2)$

$- y_{n+1} \cdot x - n y_n \cdot 1 = 0$

$\Rightarrow y_{n+2}(1-x^2) - (2n+1)x y_{n+1} - n^2 y_n = 0 \longrightarrow ④$

Put $x=0$, in ①, ②, ③ & ④ we get

$(y)_0 = 0$

$(y_1)_0 = 0$

$(y_2)_0 = 2$

$(y_{n+2})_0 = n^2 (y_n)_0 \longrightarrow ⑤$

Put $n = 1, 2, 3, 4 \ldots$ in eq^n ⑤, we get

$(y_3)_0 = 1^2 (y_1)_0 = 0$

$(y_4)_0 = 2^2 (y_2)_0 = 2^2 \cdot 2 = 2 \cdot 2^2$

$(y_5)_0 = 3^2 (y_3)_0 = 0$

$(y_6)_0 = 4^2 (y_4)_0 = 2 \cdot 2^2 \cdot 4^2$

$\vdots$

$$\boxed{(y_n)_0 = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2 \cdot 2^2 \cdot 4^2 \cdots (n-2)^2 & \text{when } n \text{ is even} \end{cases}}$$

Ex-5: If $y = (\sinh^{-1}x)^2$, prove that (AKTU-2010).

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Also find $(y_n)_0$.

Ex-6: If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1}$
$- (n^2+a^2)y_n = 0$. find $(y_n)_0$. (AKTU-204)

Ex-7: If $y = e^{m\cos^{-1}x}$, show that (AKTU-2001, 2012, 2016).

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ and Calculate $y_n(0)$.

Sol-7: Given $y = e^{m\cos^{-1}x} \longrightarrow (1)$

$\therefore \; y_1 = e^{m\cos^{-1}x} \cdot \frac{m(-1)}{\sqrt{1-x^2}} \longrightarrow (2)$

$\Rightarrow y_1\sqrt{1-x^2} = -m\, e^{m\cos^{-1}x}$

$\Rightarrow y_1\sqrt{1-x^2} = -my$ (from (1))

Squaring both side, we get

$y_1^2(1-x^2) = m^2y^2$

Again diff. above eq^n, we get

$2y_1\cdot y_2(1-x^2) + y_1^2(-2x) = m^2 2y\cdot y_1$

$\Rightarrow y_2(1-x^2) - y_1x - m^2y = 0 \longrightarrow (3)$

Diff. n times eq^n (3) by Leibnitz Theorem, we get

$D^n[y_2(1-x^2)] - D^n[y_1\cdot x] - m^2D^ny = 0$

$\Rightarrow D^ny_2\cdot(1-x^2) + {}^nC_1 D^{n-1}y_2\cdot D(1-x^2) + {}^nC_2 D^{n-2}y_2\cdot D^2(1-x^2)$
$- D^ny_1\cdot x - {}^nC_1 D^{n-1}y_1\cdot Dx - m^2D^ny = 0$

$\Rightarrow y_{n+2}\cdot(1-x^2) + n\,y_{n+1}\cdot(-2x) + \frac{n(n-1)}{\lfloor 2}y_n(-2)$
$- y_{n+1}\cdot x - n\,y_n\cdot 1 - m^2y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) - (2n+1)xy_{n+1} + (-n^2+n-n-m^2)y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \longrightarrow (4)$

Put $x=0$ in eq^n (1), (2), (3) & (4), we get

$(y)_0 = e^{m\pi/2}$

$(y_1)_0 = -m\,e^{m\pi/2}$

$(y_2)_0 = m^2(y)_0 = m^2 e^{\frac{m\pi}{2}}$

& $(y_{n+2})_0 = (n^2+m^2)(y_n)_0 \longrightarrow (5)$

Put $n = 1, 2, 3, 4, 5, \dots$ in eqn (5), we get

$$(y_3)_0 = (1^2+m^2)(y_1)_0 = (1^2+m^2)(-m)e^{\frac{m\pi}{2}} = -m(1^2+m^2)e^{\frac{m\pi}{2}}$$

$$(y_4)_0 = (2^2+m^2)(y_2)_0 = (2^2+m^2)m^2e^{\frac{m\pi}{2}} = m^2(2^2+m^2)e^{\frac{m\pi}{2}}$$

$$(y_5)_0 = (3^2+m^2)(y_3)_0 = -m(1^2+m^2)(3^2+m^2)e^{\frac{m\pi}{2}}$$

$$(y_6)_0 = (4^2+m^2)(y_4)_0 = m^2(2^2+m^2)(4^2+m^2)e^{m\pi/2}$$

$\vdots$

$$\boxed{y_n(0) = \begin{cases} -m(1^2+m^2)(3^2+m^2)\cdots[(n-2)^2+m^2]\,e^{\frac{m\pi}{2}} & \text{when } n \text{ is odd.} \\ m^2(2^2+m^2)(4^2+m^2)\cdots[(n-2)^2+m^2]\,e^{m\pi/2} & \text{when } n \text{ is even.} \end{cases}}$$

Ex-8: If $y = [x + \sqrt{1+x^2}]^m$, find $y_n(0)$. [AKTU-2013, 2018]. 2005

Ex-9: Find $y_n(0)$, when $y = \log[x + \sqrt{1+x^2}]$.

Ex-10: If $y = [\log(x+\sqrt{1+x^2})]^2$, Find all the derivatives of y w.r.t x, when $x = 0$. (AKTU-2003).

Ex-11: If $\log y = \tan^{-1}x$, show that $(1+x^2)y_{n+2} + [2(n+1)x - 1]y_{n+1} + n(n+1)y_n = 0$. and hence find y_3, y_4 and y_5 at $x=0$. (ST-2019).

Ex-12: If $y = \tan^{-1}x$, prove that

Imp $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$. find $(y_n)_0$.

(AKTU-2009)

Soln: Given $y = \tan^{-1}x \to (1)$

$\therefore\ y_1 = \frac{1}{1+x^2} \to (2)$

$\Rightarrow\ y_1(1+x^2) = 1$

on Diff. above eqn n time by Leibnitz theorem, we get

$D^n[y_1(1+x^2)] = 0$.

$\Rightarrow D^n y_1 \cdot (1+x^2) + {}^nC_1 D^{n-1} y_1 \cdot D(1+x^2) + {}^nC_2 D^{n-2} y_1 \cdot D^2(1+x^2) = 0$

$\Rightarrow y_{n+1} \cdot (1+x^2) + n\, y_n \cdot 2x + \frac{n(n-1)}{\lfloor 2} y_{n-1} \cdot 2 = 0$

$\Rightarrow y_{n+1} \cdot (1+x^2) + 2nx\, y_n + n(n-1)\, y_{n-1} = 0 \longrightarrow (3)$

Put $x=0$ in (1), (2) & (3), we get

$y(0) = 0 \quad , \quad y_1(0) = 1$

$y_{n+1}(0) = -n(n-1)\, y_{n-1}(0) \longrightarrow (4)$

Put $n = 1, 2, 3, 4, 5 \ldots$ in eq (4), we get

$y_2(0) = -1 \cdot 0 \cdot y_0(0) = 0$

$y_3(0) = -2 \cdot 1 \cdot y_1(0) = (-1)^1 \lfloor 2 = (-1)^{\frac{3-1}{2}} \lfloor 2$

$y_4(0) = -3 \cdot 2 \cdot y_2(0) = 0$

$y_5(0) = -4 \cdot 3 \cdot y_3(0) = (-1)^2 4 \cdot 3 \cdot \lfloor 2 = (-1)^2 \lfloor 4 = (-1)^{\frac{5-1}{2}} \lfloor 4$

$y_6(0) = -5 \cdot 4\, y_4(0) = 0$

$y_7(0) = -6 \cdot 5\, y_5(0) = (-1)^3 \lfloor 6 = (-1)^{\frac{7-1}{2}} \lfloor 6$

$\vdots$

$$y_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \lfloor n-1 & \text{if } n \text{ is odd} \end{cases}$$

MATHEMATICS-I
KAS-103T
Lecture No - 22

Module - II
Diff. Calculus-1
[redacted]

(1)

Asymptotes: The asymptote is a line that approaches a given curve arbitrary closely.

(1) **Asymptotes || to x-axis:** Equating to zero the coefficient of highest power of x, we get asymptote || to x-axis. If the coefficient is constant, then there is no asymptote || to x-axis.

(2) **Asymptotes || to y-axis:** Equating to zero the coefficient of highest power of y, we get asymptote || to y-axis. If the coefficient is constant, then there is no asymptote || to y-axis.

(3) **Oblique asymptote →** If there is no asymptote || to x & y axis then the curve have oblique asymptote.

Rule to find Oblique asymptote → Let $y = mx + c$ be the required asymptote. Then

Step-1: Put $y = m$ & $x = 1$ in the highest degree term & get $\phi_n(m)$, where n is the highest degree of curve.

Step-2: Put $\phi_n(m) = 0$ & find the values of $m = m_1, m_2, \ldots$

Step-3: Put $y = m$ & $x = 1$ in next degree term i.e $(n-1), (n-2) \ldots$ degree terms & get $\phi_{n-1}(m), \phi_{n-2}(m)$ etc.

Step-4: **Determination of c →** (1) If m has distinct values then use

$$\boxed{c\,\phi_n'(m) + \phi_{n-1}(m) = 0}$$

(2) If m has ~~distinct~~ same two values the use

$$\boxed{\frac{c^2}{\lfloor 2}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0}$$

Note: (1) If the given curve is of nth degree then the curve have at most n asymptote.

(2) If roots of m are imaginary then omit it.

(*) Curve Tracing ...

(Parallel asymptote)

Ex-1: find the asymptote of $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$.

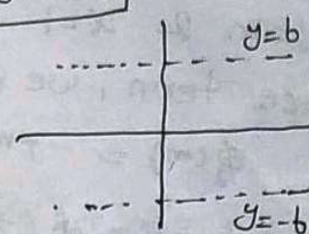
Solⁿ: The given curve $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$

$$\Rightarrow a^2y^2+b^2x^2=x^2y^2$$

$$\Rightarrow x^2y^2-a^2y^2-b^2x^2=0 \longrightarrow (1)$$

Equating to zero the coefficient of highest power of x, we get asymptote parallel to x axis,

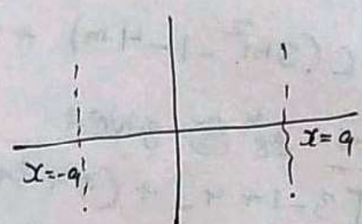
i.e $y^2-b^2=0 \Rightarrow y^2=b^2 \Rightarrow \boxed{y=\pm b}$



Equating to zero the coefficient of highest power of y, we get asymptote parallel to y axis,

i.e $x^2-a^2=0$

$\Rightarrow \boxed{x=\pm a}$



Ex-2: find the asymptote of the given curve.

Solⁿ: (1) $x^2y^2=a^2(x^2+y^2)$ (2) $\frac{a^2}{x}+\frac{b^2}{y}=1$

(3) $y^3-xy^2=x^2+1$. (4) $\frac{a^2}{x^2}-\frac{b^2}{y^2}=1$.

[Ans Asymptote ∥ to x-axis are imaginary so does not exist]

& Asymptote ∥ to y-axis is $\boxed{x=\pm a}$

Oblique asymptote:

Ex: find the asymptote of the curve

$$y^3-x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0.$$

Solⁿ: Since the coefficient of the highest power of x & y are constant then there is no asymptote ∥ to x & y axis.

Let $y = mx + c$ be the required asymptote.

Put $y = m$ & $x = 1$ in the highest degree terms, we get

$$\phi_3(m) = m^3 - m - 2m^2 + 2 \rightarrow (3)$$

Put $\phi_3(m) = 0$.

$\Rightarrow m^3 - 2m^2 - m + 2 = 0$

$\Rightarrow m^2(m-2) - 1(m-2) = 0$

$\Rightarrow (m^2-1)(m-2) = 0 \Rightarrow \boxed{m = -1, 1, 2}$

Put $y = m$ & $x = 1$ in the next degree term i.e IInd degree term, we get

$$\phi_2(m) = -7m + 3m^2 + 2 \rightarrow (4)$$

Determination of $c \rightarrow$ We have $c\,\phi_3'(m) + \phi_2(m) = 0$.

$\Rightarrow c(3m^2 - 1 - 4m) + (-7m + 3m^2 + 2) = 0 \rightarrow (5)$

for $m = -1 \rightarrow$ eq^n (5) gives

$c[3 - 1 + 4] + (7 + 3 + 2) = 0 \Rightarrow \boxed{c = -2}$

$\therefore$ Asymptote is $y = (-1)x - 2 \Rightarrow \boxed{x + y + 2 = 0}$

for $m = 1 \rightarrow$ from (5),

$c(3 - 1 - 4) + (-7 + 3 + 2) = 0$

$\Rightarrow -2c - 2 = 0 \Rightarrow \boxed{c = -1}$

$\therefore$ Asymptote is $\boxed{y = x - 1}$

for $m = 2 \rightarrow$ from (5),

$c[12 - 1 - 8) + (-14 + 12 + 2) = 0$

$\Rightarrow 3c = 0 \Rightarrow c = 0$.

Hence Asyptote is $\boxed{y = 2x}$

$\therefore$ Hence asymptote of the given curve are

$$\boxed{x + y + 2 = 0, \quad y = x - 1, \quad y = 2x}$$

Ex-37 find all the asymptote for the curve

$$(x-y)^2(x+2y-1) = 3x+y-7$$

$$\boxed{\text{Ans } y = -\frac{1}{2}x + \frac{9}{2}, \quad y = x \pm \frac{2}{\sqrt{3}}}$$

Ex-44 find asymptote of the curve

(i) $x^2y^2 - y^2 - 2 = 0$

(ii) $x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0$.

(iii) $(x^2 - y^2)^2 - 4y^2 + y = 0$

(iv) $xy^2 = 4a^2(2a-x)$ (2014)

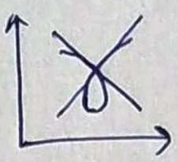
(v) $x^3 + 3xy^2 - x^2y - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 7 = 0$

(3) <u>Multiple Point</u> → A point on the curve is said to be multiple points if through this point more than one branches of a curve passes.

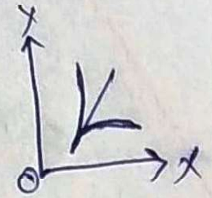
(4) Double Point → A point on the curve is called a double point if two branches of the curve passes through it.

<u>Types of double points</u> →

(1) <u>Node</u> → A double point on a curve is said to be a node, if through this double point two branches of the curve passes which are real and having two different tangents at that point.



(2) <u>Cusp</u> → A double point on a curve is called a cusp if through this double point two real branches of the curve passes and have real coincident tangents at that point.



<u>Tangent at origin</u> →

If a curve passes through the origin, then the eqⁿ of the tangent at (0,0) are obtained by equating to zero the lowest degree terms in the eqⁿ of the curve.

<u>Ex 1</u> $x^3+y^3-3axy=0 \rightarrow ①$

Clearly the curve passes through the origin, then equating to zero the lowest (2nd) degree term in the eeⁿ ①, we get tangent at origin

$-3axy=0$

$\Rightarrow x=0,\ y=0.$

Thus at the origin there are two real and distinct tangents.

Hence (0,0) is a node.

<u>Ex</u> → Examine the nature of the origin on the curve

$$(2x+y)^2-6xy(2x+y)-7x^3=0$$

~~Put 2nd degree term~~

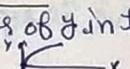
Clearly curve " " " $(2x+y)^2=0 \Rightarrow (2x+y)=0,\ 2x+y=0$.

Thus at (0,0) there are two coincident tangent. Thus (0,0) is a cusp.

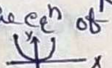
① Euler Th...

(*) Curve Tracing of cartesian curve →

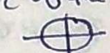
① **Symmetry** → (i) If the powers of y in the eqⁿ of the curve are all even, the curve is symmetrical about x-axis. Ex: $y^2 = 4ax$



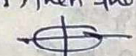
(ii) If the powers of x in the eqⁿ of the curve are all even, then the curve is symmetrical about y axis. Ex: $x^2 = 4ay$



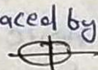
(iii) If the powers of x as well as y in the eqⁿ of the curve are all even, then the curve is symmetrical about both axes. Ex: (a) $x^2 + y^2 = a^2$



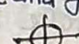
(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



(iv) If the eqⁿ of curve remains unchanged when x is replaced by $-x$ & y by $-y$, then the curve is symmetrical in opposite quadrants. Ex: $x^2 + y^2 = a^2$

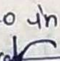


(v) If the eqⁿ of curve remains unchanged when x and y are interchanged, then the curve is symmetrical about the line $y = x$. Ex: ① $x^2 + y^2 = a^2$



(b) $\frac{x^2}{a^2}$

② **Nature of origin** → (i) Put $x=0$ & $y=0$ in the eqⁿ of curve, if satisfied then curve passes through the origin. Ex: $y^2 = 4ax$



(ii) **Tangent at origin** → find the tangent at (0,0) be equating to zero the lowest degree terms of the curve. Ex: $y^2 = 4ax$ then $x=0$ i.e. y-axis is tangent.
If there is two tangent at (0,0) then origin will be a double point & then find the nature of this double point.

③ **Asymptote** → find asymptote ‖ or oblique as above.

④ **Point of intersection** → (i) Put $x=0$ & find value of y. We get the point of intersection with the y-axis as ($x=0$ is on y-axis).
(ii) Put $y=0$ & find value of x. We get the point of intersection with the x-axis as ($y=0$ is on x-axis)
(iii) find the values of x as y goes 0 to ∞. find the values of y as x goes 0 to ∞.

⑤ **Regions** → If the value of y is imaginary for certain value of x, then curve does not exist for such values.

⑥ **Table** → Prepare a table for certain values of x & y & draw the curve passing through these points of table. (Special Points).

* Finally draw the curve keeping all the above points in your mind.

Ex-1→

Trace the curve $y^2(2a-x) = x^3$. (2015, 2006, 2004). AKTU

Sol^n→ Given $y^2(2a-x) = x^3 \longrightarrow ①$

① Symmetry→ Since the power of y in the eq^n of the curve are all even then the curve is symmetrical about x-axis.

② Nature of origin→

Put $x=0, y=0$ in the equation of the curve, the given curve is satisfied So the curve passing through the origin.

Equating to zero the lowest degree term in ①, we get

$y^2 = 0$, $\boxed{y=0,\ y=0}$ i.e x-axis.

Therefore two real & coincident tangent at origin. Hence $(0,0)$ is a cusp.

③ Asymptote→ Equating to zero the coefficient of highest power of y, we get asymptote ∥ to y-axis i.e

$$2a - x = 0 \Rightarrow \boxed{x = 2a}$$

There is no asymptote ∥ to x-axis. Also there is no oblique asymptote.

④ Point of intersection→

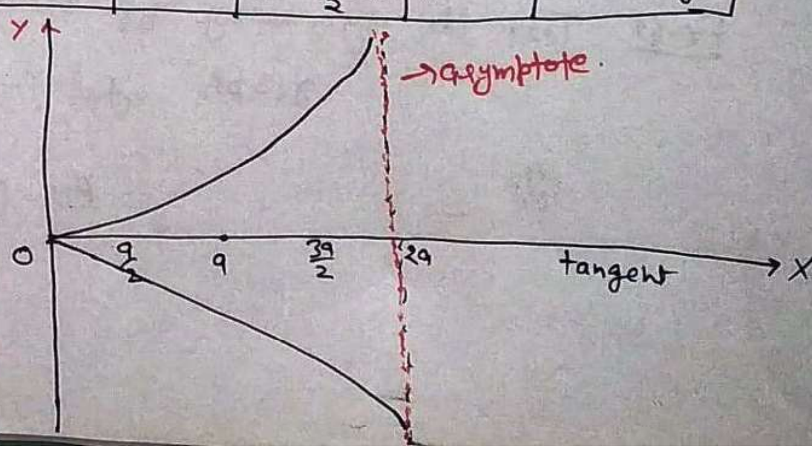
from the eq^n of the curve it is obvious that the curve does not cut the coordinate axis.

⑤ from ①, we have $y = x\sqrt{\dfrac{x}{2a-x}}$

Table→

x	$x<0$	$x=0$	$a/2 = x$	$x=a$	$x=\frac{3a}{2}$	$x=2a$	$x>2a$
y	y is imaginary	$y=0$	$y=\frac{a}{2}\sqrt{\frac{1}{3}}$	$y=a$	$y=\frac{3a}{2}\sqrt{3}$	$y=\infty$	y is imag.

Keeping all the points in ~~mid~~ mind ⟶



Q. Trace the curve : $y^2(a+x) = x^2(3a-x)$ $\to$ ① (AKTU-2014, 2018)

Sol.ⁿ ① Symmetry → Since the curve ① contains only even power of y, then the curve is symmetrical about x-axis.

② Nature of origin → Clearly the curve passes through the origin $(0,0)$. Then the tangents at the origin is obtain by equating to zero the lowest degree term in ①, we get $y^2 = 3x^2 \Rightarrow y = \pm\sqrt{3}x$

$\therefore$ we get two real & distinct tangent at $(0,0)$.

So $(0,0)$ is a node.

③ Asymptote → Equating to zero the coefficient of highest power of y, we get asymptote ∥ to y axis is $a+x=0$, $\boxed{x=-a}$

No asymptote except $x=-a$.

④ Point of intersection with axes →

Put $y=0$, we get $x^2(3a-x)=0 \Rightarrow x=0, 3a$.

$\therefore$ point of intersection with x axis are $(0,0)$, $(3a,0)$.

Put $x=0$, we get $y=0$. $\Rightarrow$ The curve will not intersect y-axis except $(0,0)$

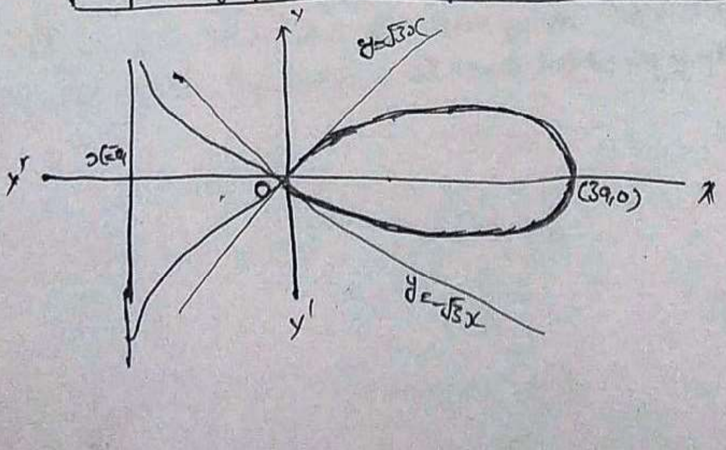
⑤ from ①, $y = x\sqrt{\dfrac{3a-x}{a+x}}$ $\to$ ②

Also $\dfrac{3a-x}{a+x} < \dfrac{3a}{a} \Rightarrow \sqrt{\dfrac{3a-x}{a+x}} < \sqrt{3} \Rightarrow x\sqrt{\dfrac{3a-x}{a+x}} < \sqrt{3}x \Rightarrow y < \sqrt{3}x$.

$\therefore$ Curve lies below tangent $y=\sqrt{3}x$

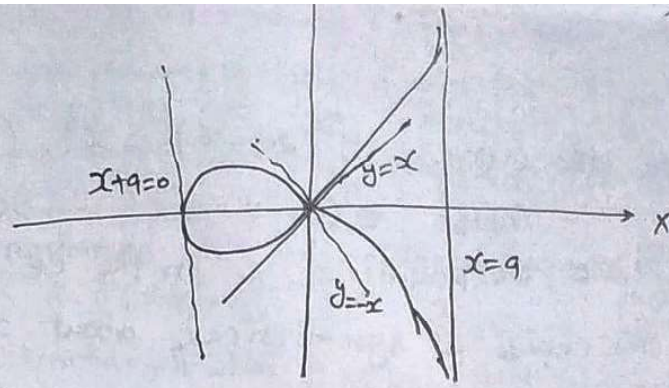
⑥ Table →

x	$x<-a$	$x=-a$	$x=0$	$x=\frac{a}{2}$	$x=a$	$x=2a$	$x=3a$	$x>3a$
y	y is imag.	∞	0	$\frac{a}{2}\sqrt{\frac{5}{3}}$	a	$\frac{2a}{\sqrt{3}}$	0	y is imaginary



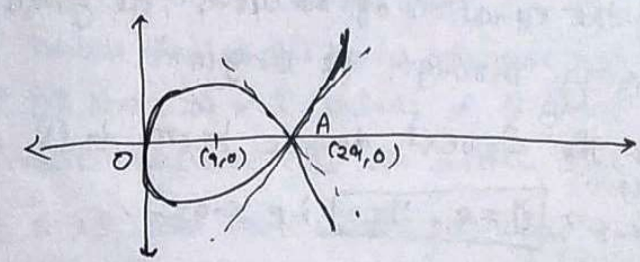
Ex-2 → Trace the curve
$$y^2(a-x) = x^2(a+x)$$

Ans



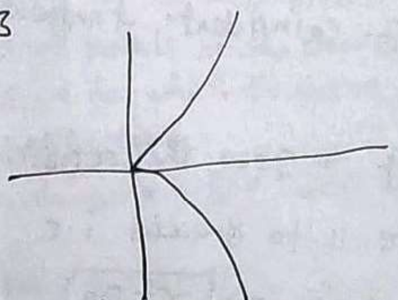
Ex-3 → Trace $4ay^2 = x(x-2a)^2$ (AKTU-2017)

Ans →



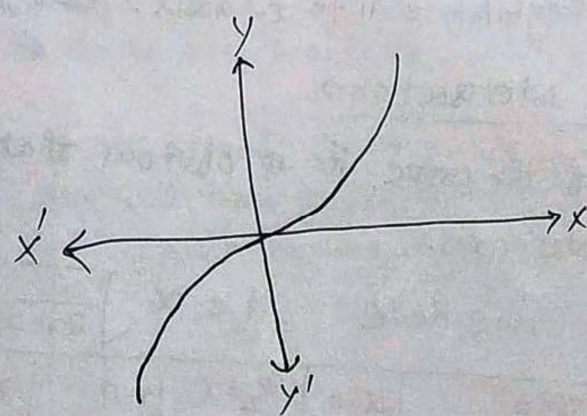
Ex-4 → Trace $y^2 = x^3$

Ans



Ex-5 → Trace $y = x^3$

Ans →



Ex-6 → Test the symmetry of the graph $y = 1 - \frac{1}{1+x^2}$. (ST-I 2020).

Ans (Symmetrical about y-axis).

MATHEMATICS-I
KAS-103T

Lecture No - 23

Module-II
Diff. Calculus-I
[redacted]

Curve Tracing in Polar Coordinates → If the curve is $r=f(\theta)$. Then ②

① Symmetry → (i) If the curve $r=f(\theta)$ remains unchanged when θ is replaced by $-\theta$, then the curve is symmetrical about the initial line. Ex $r^2=a\cos\theta$, $r=a\cos 2\theta$.

(ii) If the curve $r=f(\theta)$ remains unchanged when r is replaced by $-r$, then the curve is symmetrical about the pole (origin). Ex $r^2=a\cos 2\theta$.

(iii) If the curve $r=f(\theta)$ remains unchanged when r is replaced by $-r$ & θ by $-\theta$, then the curve is symmetrical about $\theta=\frac{\pi}{2}$ (half ray). Ex $r^2=a\cos 2\theta$.

(iv) If the curve $r=f(\theta)$ remains unchanged when θ is replaced by $(\pi-\theta)$, then the curve is symmetrical about $\theta=\frac{\pi}{2}$. Ex $r=a(1+\sin\theta)$ & $r=a\sin\theta$

② Nature of Pole or Origin → Put $r=0$ in the given curve, if we get some real values of θ, then the curve passes through the pole otherwise not. Ex $r=a(1-\cos\theta)$.

③ Tangent at Pole → Put $r=0$, the real values of θ gives tangent at pole.

④ Find the points, where the curve cuts the initial line $(\theta=0)$ and the line $\theta=\frac{\pi}{2}$.

⑤ Asymptotes → If $\theta\to\theta_1$ (some finite ~~value~~ value) when $r\to\infty$, then $\theta=\theta_1$ ~~there~~ is an asymptote. Then find all the asymptote.

⑥ Point of intersection → Find some points on the curve for possible values of θ. (especially for values of $\theta=\alpha$ for which the curve is symmetrical).

⑦ Regions where the curve does not exist →

(i) If we obtain the values of r, imaginary for $\alpha<\theta<\beta$, then the curve will not exist in this region.

Ex If $r^2=a^2\cos 2\theta$, then for $\frac{\pi}{4}<\theta<\frac{3\pi}{4}$, $\cos 2\theta$ is negative $\Rightarrow r^2$ is $-ve$ $\Rightarrow r$ is imaginary. Hence this curve does not exist b/w $\theta=\frac{\pi}{4}$ to $\theta=\frac{3\pi}{4}$.

(ii) If the ~~greatest~~ & least numerical values of r be respectively a & b, then the curve lies entirely within the circle $r=a$ & entirely out-side the circle $r=b$.

⑧ Construct the Table →

Find r when θ varies in the interval $(0,\infty)$ & $(-\infty,0)$ with maximum & minimum values of r. Then plot all these points. (Special Points)

⑨ Draw the graph, by keeping all above points in your mind.

(7) Trace the curve $r^2 = a^2 \cos 2\theta$. $\boxed{\text{(AKTU - 2012, 2018)}}$ $\boxed{2020, 2015}$
(Lemniscate) → (1)

Sol^n → (1) Symmetry → The curve is symmetrical about the initial line, the line $\theta = \frac{\pi}{2}$, and pole

(2) Nature of origin → Put $r=0$ in (1), we get

$\cos 2\theta = 0 = \cos(\pm\frac{\pi}{2}) \Rightarrow \theta = \pm\frac{\pi}{2}$ which are real

Hence the curve passes through the pole & the tangents at pole are $\theta = \pm\frac{\pi}{4}$.

~~Also, when~~

(3) Asymptote → Since $|\cos 2\theta| \le 1$ then for no values of θ, $r \to \infty$

∴ Curve has no asymptote.

(4) Point of intersection → When $\theta = 0$, $r = \pm a$.

Hence, the curve meets the initial line at $(a,0)$ & $(-a,0)$.

(5) ~~Region~~

If $r^2 = a^2 \cos 2\theta$, the for $\frac{\pi}{4} < \theta < \frac{3\pi}{4} \Rightarrow \cos 2\theta$ is $-ve$.

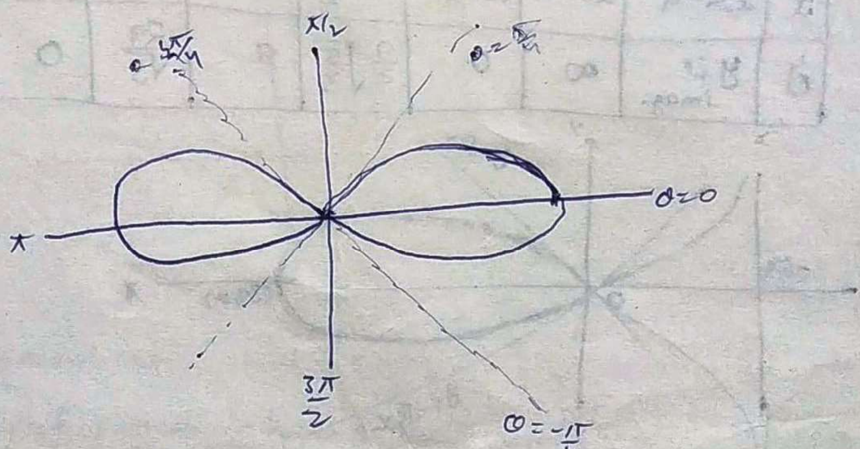
$\Rightarrow r^2$ is $-ve \Rightarrow r$ is imaginary,

then for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, the curve (1) does not exist.

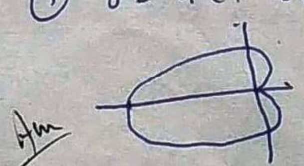
(6) Table →

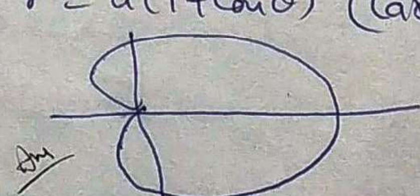
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{\pi}{4} < \theta < \frac{3\pi}{4}$
r	a	$\pm\frac{a}{2}$	0	0	$-a$	imag.

Hence



Q → Trace (1) $r = a(1 - \cos\theta)$ (2) $r = a(1 + \cos\theta)$ (Cardioids)





Ex- Trace the curve $r = a(1+\cos\theta)$ (UPTU - 2008, 2013).
(Cardioid)

Sol^n (1) **Symmetry** - Put $\theta = -\theta$, no change occurs. Then the curve is symmetrical about initial line.

(2) **Pole** :- Put $r=0$, we get $a(1+\cos\theta) = 0 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi, \pm 3\pi$.
because real values of θ exists. Then curve passes through pole.

(3) **Tangent at Pole** :- At $r=0$, $\theta = \pm\pi, \pm 3\pi$ are tangents to the curve at pole.

(4) **Asymptote** :- $r = a(1+\cos\theta)$, since $|\cos\theta| \le 1$, for all values of θ then r can never tends to ∞.
$\therefore$ No asymptote exists.

(5) **Region** :- $r = a(1+\cos\theta)$ since $|\cos\theta| \le 1$

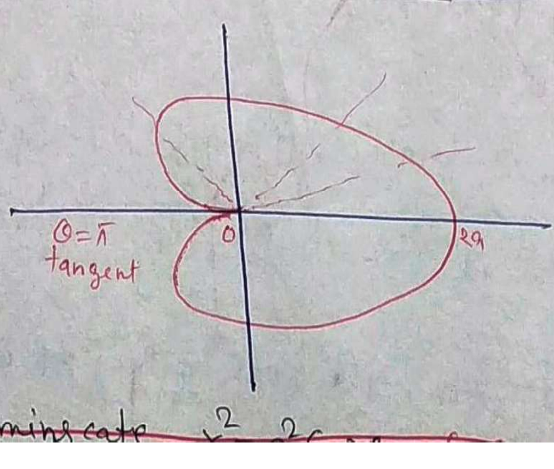
$\theta = \pm\pi$ when $r=0$ $\therefore$ $-1 \le \cos\theta \le 1$.

$\theta = 0$ when $r = 2a$.

$\therefore$ curve varies in range $0 \le r \le 2a$.

(6) **Table** :-

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π
r	$2a$		a		0	$-a$	0



~~Ex- Trace Leminscate~~ r^2 ...

MATHEMATICS-I
KAS-103T
Lecture No- 24

Module-III
Differential Calculus-II
By Dr. Anuj Kumar

Partial differentiation →

Consider a funcⁿ z of two or more independent variables, then partial derivative of this funcⁿ z w.r.t any one of the independent variables is the ordinary derivative of z w.r.t that variable, treating all other variables as constant.

Let $z = f(x,y)$.

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x \longrightarrow p$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y \longrightarrow q$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} \longrightarrow r$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \longrightarrow s$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy} \longrightarrow t.$$

Note
$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

Ex-1 Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ for the funcⁿ

$$z = ax^2 + 2hxy + by^2.$$

Solⁿ Given $z = ax^2 + 2hxy + by^2 \longrightarrow ①$

then $z_x = \frac{\partial z}{\partial x} = 2ax + 2hy \rightarrow ②$

$z_y = \frac{\partial^2}{\partial y} = 2hx + 2by \rightarrow ③$

from ②,

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = 2a$$

$$z_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2h$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = 2b.$$

Ex-2: If $z = f(x+ct) + \phi(x-ct)$ show that
$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$ (AKTU-2011)

Solⁿ: Given $z = f(x+ct) + \phi(x-ct) \rightarrow (1)$

from (1), $$\frac{\partial z}{\partial x} = f'(x+ct)\cdot 1 + \phi'(x-ct)\cdot 1$$

& $$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \rightarrow (2)$$

Again from (1),
$$\frac{\partial z}{\partial t} = f'(x+ct)\cdot c + \phi'(x-ct)\cdot(-c)$$
$$\frac{\partial^2 z}{\partial t^2} = f''(x+ct)\cdot c^2 + \phi''(x-ct)\cdot(-c)^2$$
$$\Rightarrow \frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + \phi''(x-ct)\right]$$
$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}}$$ from (2).

Short Questions

(1) find the first order partial derivatives of

i) $u = y^x$ ii) $u = \log(x^2+y^2)$.

(2) If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

(3) If $f(x,y) = x^3y - xy^3$, find $\left[\frac{1}{\frac{\partial f}{\partial x}} + \frac{1}{\frac{\partial f}{\partial y}}\right]_{\substack{x=1\\y=2}}$ (AKTU-2010) (Ans- $-\frac{13}{22}$)

(4) If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$
where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$. (AKTU-2015).

(5) If $y = e^{ax+by} f(ax-by)$, show that
$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

(6) If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$, show that
$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$ [AKTU-2012].

Sol^n 6: Given $u(x,y,z) = \log(\tan x + \tan y + \tan z) \longrightarrow ①$

Differentiating eq^n ① partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x) \longrightarrow ②$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z}\cdot(\sec^2 y) \longrightarrow ③$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z}\cdot(\sec^2 z) \longrightarrow ④$$

Now

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \cdot \frac{\partial u}{\partial z}$$

$$= \frac{(2\sin x\cos x)\sec^2 x + (2\sin y\cos y)\cdot\sec^2 y + (2\sin z\cos z)\cdot\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$$

$$= 2.$$

$$\Rightarrow \boxed{\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2}$$ Hence proved.

(Long Question)

Ex-7: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$ [AKTU-2011, 2015]

Sol^n: Given $u = \log(x^3 + y^3 + z^3 - 3xyz) \longrightarrow ①$

Diff. eq^n ① w.r.t x partially, we get

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times (3x^2 - 3yz)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz} \longrightarrow ②$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3(y^2 - zx)}{x^3 + y^3 + z^3 - 3xyz} \longrightarrow ③$$

& $$\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz} \longrightarrow ④$$

Adding eq^n (2), (3) & (4), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z} \longrightarrow (5)$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) u.$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot\frac{3}{x+y+z} \quad \text{(from 5)}$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}}$$ Proved.

Ex-8: If $x^x y^y z^z = c$, show that $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.

(AKTU-2013, 2016)

Sol^n: Given $x^x y^y z^z = c$ → (1), take z as dependent variable.

Taking log, $x\log x + y\log y + z\log z = \log c \longrightarrow (2)$

Diff. eq^n (2) w.r.t x, partially, we get

$$\left(x\cdot\frac{1}{x}+\log x\right)+\left(z\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}+\log z\,\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow (1+\log x)+(1+\log z)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{1+\log z} \longrightarrow (3)$$

Similarly, $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \longrightarrow (4)$

Diff. eq^n (4) partially w.r.t x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log y)\frac{\partial}{\partial x}\frac{1}{(1+\log z)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = -(1+\log y) \cdot \frac{(-1)}{(1+\log z)^2} \cdot \frac{1}{z} \frac{\partial z}{\partial x}$$

$$= \frac{(1+\log y)}{(1+\log z)^2} \cdot \frac{1}{z} \cdot \frac{[-(1+\log x)]}{[1+\log z]} \quad [\text{from-③}]$$

At $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)}{(1+\log x)^2} \cdot \frac{1}{x} \frac{(1+\log x)}{(1+\log x)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(\log e + \log x)}$$

$$= -\frac{1}{x \log ex}$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}}.$$

Ex-9: If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$ [AKTU-2016, 2014]

Solⁿ: Given $u = f(r) \longrightarrow ①$

& $r^2 = x^2 + y^2 \longrightarrow ②$

from ②, $2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ $\Bigg\} \longrightarrow ③$

from ①, $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r} \longrightarrow ④$ [from ③]

Diff. again with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = f'(r)\left[\frac{1}{r} \cdot 1 + x \cdot \frac{(-1)}{r^2} \frac{\partial r}{\partial x}\right] + \frac{x}{r} \cdot f''(r) \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \longrightarrow ⑤ \quad [\text{from ③}]$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r) \longrightarrow ⑥$$

Adding ⑤ & ⑥, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{r}f'(r)-\frac{(x^2+y^2)}{r^3}f'(r)+\frac{(x^2+y^2)}{r^2}f''(r)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{r}f'(r)-\frac{1}{r}f'(r)+f''(r) \quad (\text{from } ②)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)}$$ Hence proved.

Ex-10: If $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$, find the value of $\frac{\partial^2 u}{\partial x\partial y}$.

(AKTU-2018, 2015)

Sol^n: Given $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y} \longrightarrow ①$

Diff. eq^n ① w.r.t y, partially, we get

$$\frac{\partial u}{\partial y}=x^2\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}-y^2\cdot\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{-x}{y^2}\right)-2y\tan^{-1}\frac{x}{y}$$

$$\Rightarrow \frac{\partial u}{\partial y}=\frac{x^3}{x^2+y^2}+\frac{xy^2}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}.$$

$$\Rightarrow \frac{\partial u}{\partial y}=\frac{x(x^2+y^2)}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}$$

$$\Rightarrow \frac{\partial u}{\partial y}=x-2y\tan^{-1}\frac{x}{y} \longrightarrow ②$$

Diff eq^n ② w.r.t x, partially, we get

$$\frac{\partial^2 u}{\partial x\partial y}=1-2y\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x\partial y}=1-\frac{2y^2}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x\partial y}=\frac{(x^2+y^2)-2y^2}{x^2+y^2} \Rightarrow \boxed{\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}}$$

Ex-10: If $e^{-\frac{z}{x^2+y^2}}=(x-y)$ then show that $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=x^2-y^2$.

(AKTU-2017)

Ex-11: If $u=(x^2+y^2+z^2)^{-1/2}$ show that $xu_x+yu_y+zu_z=-u$.

(AKTU-2016)

Ex-12: If $u=\log\sqrt{x^2+y^2+z^2}$ show that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1.$$ (AKTU-2011)

Ex-13: If $z(x+y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \quad \text{(AKTU-2010)}$$

Ex-14: Find the value of n so that the equation $V = r^n (3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0. \quad \text{(AKTU-2014)}$$

* Special Examples *

① If $u = \tan^{-1}\frac{xy}{\sqrt{1+x^2+y^2}}$, show that $\frac{\partial^2 u}{\partial x \partial y} = (1+x^2+y^2)^{-3/2}$.

② If $u = e^{xyz}$, prove $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$.

③ If $x^2 = au + bv$, $y^2 = au - bv$, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u \quad \text{[AKTU-20}$$

Sol^n: Given $x^2 = au + bv \rightarrow ①$

$y^2 = au - bv \rightarrow ②$

Diff eq^n ① partially w.r.t u keeping v constant,

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x} \rightarrow ③$$

Diff. eq^n ② partially w.r.t v keeping u constant,

$$2y\left(\frac{\partial y}{\partial u}\right)_v = -b \Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y} \rightarrow ④$$

From ① & ②, $u = \frac{x^2+y^2}{2a} \rightarrow ⑤$

Diff. eq^n ⑤ partially w.r.t x keeping y constant.

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a} \rightarrow ⑥$$

Also from ① & ②, $v = \frac{x^2-y^2}{2b} \rightarrow ⑦$. Diff. eq^n ⑦ partially w.r.t y keeping x constant, $\left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$.

Hence $\boxed{\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a}\cdot\frac{a}{2x} = \frac{1}{2}}$

& $\boxed{\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = -\frac{y}{b} \times \frac{-b}{2y} = \frac{1}{2}}$

Hence proved.

MATHEMATICS-I

* KAS-103T *

Module - III

Diff. Calculus - II

[redacted]

Lecture No- 25

Homogeneous func^n →

A func^n $f(x,y)$ is said to be homogeneous func^n of degree 'n', if each term having equal degree 'n' for the variables x & y.

Let $u = f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$

$$= x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \frac{y^2}{x^2} + \dots + a_n \frac{y^n}{x^n}\right]$$

$$= x^n f\left(\frac{y}{x}\right) \quad \text{or} \quad y^n f\left(\frac{x}{y}\right)$$

is called Homogeneous func^n of degree n.

Note→ Degree of homogeneous func^n can be obtained by replacing x by tx and y by ty.

i.e $f(tx,ty) = t^n f(x,y)$. Then n is the degree of Hom. func^n.

State and Prove Euler's Theorem on Homogeneous functions → (AKTU-2013, 2011)

Statement→

If u is a homogeneous func^n of degree n in x & y, then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{or} \quad x u_x + y u_y = nu.}$$

Proof→ Since u is a homogeneous func^n of degree n in x & y, then

$$u = x^n f\left(\frac{y}{x}\right) \longrightarrow (*)$$

Then, $\frac{\partial u}{\partial x} = n x^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\frac{(-y)}{x^2}$

$$\Rightarrow x\frac{\partial u}{\partial x} = n x^n f\left(\frac{y}{x}\right) - y x^{n-1} f'\left(\frac{y}{x}\right) \longrightarrow (1)$$

Also, $\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$

$$\Rightarrow y\frac{\partial u}{\partial y} = y x^{n-1} f'\left(\frac{y}{x}\right) \longrightarrow (2)$$

Adding (1) & (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n x^n f\left(\frac{y}{x}\right) = nu \quad (\text{from } *)$$

$$\Rightarrow \boxed{x u_x + y u_y = nu}$$ Hence proved.

(Type - I)

Note-1 → If u is a homogeneous func^n of degree n in x & y, then

$$\boxed{\begin{array}{l}(i)\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu\\ (ii)\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.\end{array}}$$

Note-2 → If u is a homogeneous func^n of degree n in x, y & z

then $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}$ or $\boxed{xu_x+yu_y+zu_z=nu}$

V. Im.

Note-3 → (i) If Verify is given in question then solve directly by previous method and use Euler's also.

(ii) If evaluate, find, prove that, show that is given in the question then use only Euler's Theorem.

Ex-1 → Verify Euler's theorem for the following functions:

(i) $f(x,y)=ax^2+2hxy+by^2$ (ii) $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$

(iii) $u=\frac{x(x^4-y^4)}{x^4+y^4}$ (AKTU-2014) (v) $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$

(iv) $z=\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$ (AKTU - 2016, 2018)

Sol^n → 1(iv) → Given $z=\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$ → ①

Now $z(tx,ty)=\frac{(tx)^{1/3}+(ty)^{1/3}}{(tx)^{1/2}+(ty)^{1/2}}=\frac{t^{1/3}}{t^{1/2}}\,\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$

$\Rightarrow z(tx,ty)=t^{-1/6}\,z(x,y)$.

⇒ z is a Homogeneous func^n of degree n, then by Euler's Theorem, we get

$$\boxed{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{6}z}\longrightarrow ②$$

Now taking logarithm both side of eq^n ①, we get

$\log z=\log(x^{1/3}+y^{1/3})-\log(x^{1/2}+y^{1/2})$

$\therefore \frac{1}{z}\frac{\partial z}{\partial x}=\frac{1}{x^{1/3}+y^{1/3}}\times\frac{1}{3}x^{-2/3}-\frac{1}{x^{1/2}+y^{1/2}}\times\frac{1}{2}x^{-1/2}$

$$\Rightarrow \frac{x}{z}\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{1/3}}{x^{1/3}+y^{1/3}} - \frac{\frac{1}{2}x^{1/2}}{x^{1/2}+y^{1/2}} \longrightarrow (3)$$

Also, $\frac{1}{z}\frac{\partial z}{\partial y} = \frac{1}{x^{1/3}+y^{1/3}} \times \frac{1}{3}y^{-2/3} - \frac{1}{x^{1/2}+y^{1/2}} \times \frac{1}{2}y^{-1/2}$

$$\Rightarrow \frac{y}{z}\frac{\partial z}{\partial y} = \frac{\frac{1}{3}y^{1/3}}{x^{1/3}+y^{1/3}} - \frac{\frac{1}{2}y^{1/2}}{x^{1/2}+y^{1/2}} \longrightarrow (4)$$

Adding (3) & (4), we get

$$\frac{x}{z}\frac{\partial z}{\partial x} + \frac{y}{z}\frac{\partial z}{\partial y} = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

$$\Rightarrow \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{6}z} \longrightarrow (5)$$

from (2) & (5), Euler's Theorem Verified.

Ex-2: If $u = (x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})$, apply Euler's theorem to find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. (AKTU - 2011)

Sol^n: Given $u = (x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})$

Then $u(tx, ty) = (t^{1/4}x^{1/4} + t^{1/4}y^{1/4})(t^{1/5}x^{1/5} + t^{1/5}y^{1/5})$

$$= t^{1/4}\cdot t^{1/5}(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})$$

$$= t^{9/20}\, u(x,y)$$

$$\Rightarrow u(tx,ty) = t^{9/20}\, u(x,y)$$

$\Rightarrow u$ is a homogeneous func^n of degree $\frac{9}{20}$. Then by Euler's Theorem

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{9}{20}u}$$

Ex-3: If $u = x^3+y^3+z^3+3xyz$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u$. (AKTU - 2010)

Sol^n: Given $u(x,y,z) = x^3+y^3+z^3+3xyz$

Then $u(tx,ty,tz) = t^3x^3 + t^3y^3 + t^3z^3 + 3t^3xyz$

$$= t^3(x^3+y^3+z^3+3xyz)$$

$$\Rightarrow u(tx,ty,tz) = t^3 u(x,y,z)$$

$\Rightarrow u(tx,ty,tz) = t^3u(x,y,z)$. Then u is a Homo. func^n of degree 3 then by Euler's Theorem

$$\boxed{x u_x + y u_y + z u_z = 3u}$$

Ex-4: If $V = \frac{x^3y^3}{x^3+y^3}$ then prove that $xV_x + yV_y = 3V$

& $x^2V_{xx} + 2xyV_{xy} + y^2V_{yy} = 6V$.

Ex-5: If $u(x,y) = (\sqrt{x}+\sqrt{y})^5$, find the value of $xu_x + yu_y$

and $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy}$. (AKTU-2013).

Soln: Given $u(x,y) = (\sqrt{x}+\sqrt{y})^5$

Now $u(tx,ty) = (\sqrt{tx}+\sqrt{ty})^5 = (\sqrt{t})^5(\sqrt{x}+\sqrt{y})^5$

$\Rightarrow u(tx,ty) = t^{5/2}\,u(x,y)$

$\therefore$ $u(x,y)$ is a Homogeneous func^n of degree $\frac{5}{2}$. Then

by Euler's theorem we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \Rightarrow \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}u}$$

$$\& \; x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$= \frac{5}{2}\left(\frac{5}{2}-1\right)u$$

$$\Rightarrow \boxed{x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = \frac{15}{4}u}$$

Ex-6: If $u = x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$, find $xu_x + yu_y$ - (AKTU-2018)

Ex-7: If $u = f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. (AKTU-2011)

Ex-8: If $u = xf\left(\frac{y}{x}\right)$, prove that $xu_x + yu_y = 3u$.

Ex-9: If $z = xyf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

Ex-10: If $u = (x^2+y^2)^{1/3}$, prove that $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = -\frac{2}{9}u$.

Ex-11: If $u = x\sin^{-1}\frac{x}{y} + y\sin^{-1}\frac{y}{x}$, evaluate $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$.
(AKTU-2010, 2015)

Ex-12: If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

(AKTU-2011, 2014).

Sol^n: 12: Given $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right) \longrightarrow (1)$

Now $u(tx, ty) = t^2x^2 \tan^{-1}\left(\frac{ty}{tx}\right) - t^2y^2 \tan^{-1}\left(\frac{tx}{ty}\right)$

$\Rightarrow u(tx, ty) = t^2\left[x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}\right]$

$\Rightarrow u(tx, ty) = t^2\, u(x, y)$

Then u is a Homogeneous func^n of degree 2, Then by Euler's Theorem for Homogeneous func^n, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \quad \Rightarrow \boxed{x u_x + y u_y = 2u}$$

& $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$
$$= 2(2-1)u$$
$$= 2u.$$

$$\Rightarrow \boxed{x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 2u}$$

Type - II (v.v. Imp).

Deductions from Euler's Theorem:

If $F(u) = V(x, y)$, where V is a homogeneous func^n in x and y of degree n, then

(i) $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)}$$

(ii) $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)\left[\phi'(u) - 1\right]$$

where $\phi(u) = n\frac{F(u)}{F'(u)}$

Ex-1: Verify Euler's theorem for the following functions:

(i) $u = \log\left(\frac{x^2+y^2}{xy}\right)$ (AKTU-2011) (ii) $u = \log\left(\frac{x^4+y^4}{x+y}\right)$ (AKTU - 2015)

Sol^n: 1(ii): Given $u = \log\left(\frac{x^4+y^4}{x+y}\right)$

$$\Rightarrow e^u = \frac{x^4+y^4}{x+y} \quad \Rightarrow F(u) = V(x, y)$$

where $F(u) = e^u$ & $V(x, y) = \frac{x^4+y^4}{x+y}$ $\Big\} \longrightarrow (1)$

Now $V(tx, ty) = \dfrac{t^4x^4 + t^4y^4}{tx + ty} = \dfrac{t^4}{t} \dfrac{x^4+y^4}{x+y}$

$\Rightarrow V(tx, ty) = t^3 V(x, y)$

Then $F(u) = V(x,y)$ & $V(x,y)$ is a Homogeneous func^n of degree 3 the by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{e^u}{e^u} = 3.$$

$$\Rightarrow \boxed{x u_x + y u_y = 3} \rightarrow (2)$$

Now, $u = \log \dfrac{x^4+y^4}{x+y}$

$\Rightarrow u = \log(x^4+y^4) - \log(x+y)$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{1}{x^4+y^4} \cdot 4x^3 - \dfrac{1}{x+y} \cdot 1$

$\Rightarrow x\dfrac{\partial u}{\partial x} = \dfrac{4x^4}{x^4+y^4} - \dfrac{x}{x+y} \rightarrow (3)$

Also, $\dfrac{\partial u}{\partial y} = \dfrac{1}{x^4+y^4} \cdot 4y^3 - \dfrac{1}{x+y} \cdot 1$

$\Rightarrow y\dfrac{\partial u}{\partial y} = \dfrac{4y^4}{x^4+y^4} - \dfrac{y}{x+y} \rightarrow (4)$

Adding eq^n (3) & (4) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4\frac{(x^4+y^4)}{x^4+y^4} - \frac{(x+y)}{x+y} = 4 - 1 = 3.$$

$$\Rightarrow \boxed{x u_x + y u_y = 3} \longrightarrow (5)$$

From (2) & (5), Euler's Theorem Verified.

Ex-2: If $u = \sec^{-1}\left(\dfrac{x^3-y^3}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\cot u$

Also evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. (AKTU-2018)

Sol^n: Given $u = \sec^{-1}\left(\dfrac{x^3-y^3}{x+y}\right)$

$\Rightarrow \sec u = \dfrac{x^3-y^3}{x+y} \Rightarrow F(u) = V(x,y)$

where $F(u) = \sec u$ & $V(x,y) = \dfrac{x^3-y^3}{x+y}$ $\Big\} \rightarrow (1)$

Now $V(tx, ty) = \dfrac{t^3}{t} \dfrac{x^3-y^3}{x+y} = t^2 V(x,y)$.

$\therefore F(u) = V$ is a Homogeneous func^n of degree 2 then by E...

theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{F(u)}{f'(u)} = 2\frac{\sec u}{\sec u \cdot \tan u} \qquad [\because F'(u)=\sec u\cdot\tan u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\cot u}$$

Also, $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)\left[\phi'(u)-1\right]$

where $\phi(u)=n\frac{f(u)}{f'(u)}$.

$$\Rightarrow \phi(u)=2\cot u$$

$$\& \quad \phi'(u)=-2\operatorname{cosec}^2 u$$

Hence $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cot u\left[-2\operatorname{cosec}^2 u-1\right]$.

$$\Rightarrow \boxed{x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy} = -2\cot u\,(2\operatorname{cosec}^2 u-1)}$$

Ex-3: If $u=\sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$ then evaluate $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$. (AKTU-2017).

(Ans $\frac{1}{144}\tan u\,(\sec^2 u-12)$. (2009)

Ex-4: If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, prove that $xu_x+yu_y=\tan u$.

Ex-5: If $u=\sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$, show that $xu_x+yu_y=-\frac{1}{12}\tan u$

Also, find $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$. (AKTU-2008, 2012).

Ex-6: If $u=\operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(13+\tan^2 u).$$

Ex-7: Prove that $xu_x+yu_y=\frac{5}{2}\tan u$, if $u=\sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$.

(AKTU-2015)

Ex-8: If $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, find $x^2u_{xx}+2xy\,u_{xy}+y^2u_{yy}$.

Ex-9: If $u=\cot^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$. (AKTU-2004, 2010)

Ex-10: Show that $xu_x+yu_y=2u\log u$, where $\log u=\frac{x^3+y^3}{3x+4y}$ (AKTU-2011)

Ex-11: If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $xu_x + yu_y = 3$. (AKTU-2012)

Ex-12: If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

(i) $xu_x + yu_y = \frac{1}{2}\tan u$ (ii) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\frac{\sin u \cos 2u}{4\cos^3 u}$.

Ex-13: If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$

$= \sin 4u - \sin 2u$.

(Type-III)

Note: If $f(u) = V(x,y,z)$ where $V(x,y,z)$ is a Homogeneous func^n of degree n then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}}$$

Ex-1: Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$, where

$u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$ (AKTU-2018).

Sol^n: Given $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right) \Rightarrow \sin u = \frac{x^3+y^3+z^3}{ax+by+cz}$

$\Rightarrow f(u) = V(x,y,z)$

where $f(u) = \sin u$ & $V(x,y,z) = \frac{x^3+y^3+z^3}{ax+by+cz}$.

Now $V(tx,ty,tz) = \frac{t^3x^3+t^3y^3+t^3z^3}{atx+bty+ctz}$

$= \frac{t^3}{t}\frac{x^3+y^3+z^3}{ax+by+cz}$

$\Rightarrow V(tx,ty,tz) = t^2 V(x,y,z)$

Then $f(u) = V(x,y,z)$ is a Homogeneous func^n of degree 2, then by Euler's Theorem

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)} = 2\cdot\frac{\sin u}{\cos u} = 2\tan u$

$\Rightarrow \boxed{xu_x + yu_y + zu_z = 2\tan u}$

Ex-2: If $u = \cos\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, prove that $xu_x + yu_y + zu_z = 0$.

Ex-3: If $u = \sin^{-1}\left(\frac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0 \cdot$$ (AKTU-2012, 2010, 2016)

Ex-4: Show that

$$xu_x + yu_y + zu_z = -2\cot u$$

where $u = \cos^{-1}\left(\frac{x^3 + y^3 + z^3}{ax + by + cz}\right)$ (AKTU-2014).

Ex-5: If $u = \log\left(\frac{x^5 + y^5 + z^5}{x^2 + y^2 + z^2}\right)$

, find $xu_x + yu_y + zu_z$.

MATHEMATICS-I
* KAS-103T *

Lecture No - 26

Module-III
Diff. Calculus- II
By Dr. Anuj Kumar

Total derivative → In partial differentiation of a func^n, only one of the independent variable varies at a time. But in total diff. all the independent variables varies simultaneously.

Composite func^n →

(i) If $u = f(x,y)$ where $x = \phi_1(t)$, $y = \phi_2(t)$ then u is called a composite func^n of t and we can find $\frac{du}{dt}$.

(ii) If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$, then z is called a composite func^n of u and v so that we can find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$.

Differentiation of composite func^n →

Case-1 → If u is composite func^n of t, defined by the relations

$u = f(x,y)$

$x = \phi_1(t)$

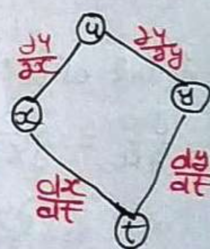
$y = \phi_2(t)$

then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}}$$

or

$$\boxed{du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy}$$



Note → If $u = f(x,y,z)$ & $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$ then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}}$$

Ex-1 → Find $\frac{du}{dt}$ if $u = x^3 + y^3$, $x = a\cos t$, $y = b\sin t$. (AKTU-2018) (2012)

Sol^n → Given $u = x^3 + y^3 \longrightarrow (1)$

then $\frac{\partial u}{\partial x} = 3x^2$, $\frac{\partial u}{\partial y} = 3y^2 \Big\} \rightarrow (2)$

& $x = a\cos t$, $y = b\sin t \Big\} \rightarrow (3)$

$\Rightarrow \frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = b\cos t \Big\} \rightarrow (4)$

Now, we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= 3x^2 \times -a\sin t + 3y^2 \times b\cos t$$

Put $x = a\cos t$, $y = b\sin t$ we get $\boxed{\frac{du}{dt} = -3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t}$

Ex-2: find $\frac{du}{dt}$, if $u = x^3y^2 + x^2y^3$ where $x = 2at^2$, $y = 2at$.

Ex-3: If $u = x^2+y^2+z^2$ & $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$ then find $\frac{du}{dt}$. (AKTU-2013, 2015).

Case-II: If $z = f(x,y)$ where $x = \phi_1(u,v)$ & $y = \phi_2(u,v)$

then
$$\boxed{\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\end{aligned}}$$

Ex-1: If $w = f(x,y)$, $x = e^u\cos v$, $y = e^u\sin v$, Show that
$$y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = e^{2u}\frac{\partial w}{\partial y}. \quad \text{(AKTU-2008)}.$$

Sol^n: Given $w = f(x,y)$

where $x = e^u\cos v$

$y = e^u\sin v$

By total differentiation, we have
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$
$$= \frac{\partial w}{\partial x}\cdot e^u\cos v + \frac{\partial w}{\partial y}e^u\sin v \longrightarrow (1)$$

& $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$
$$= \frac{\partial w}{\partial x}\cdot(-e^u\sin v) + \frac{\partial w}{\partial y}(e^u\cos v) \longrightarrow (2)$$

Now $$y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = e^u\sin v\left[\frac{\partial w}{\partial x}\cdot e^u\cos v + \frac{\partial w}{\partial y}e^u\sin v\right] + e^u\cos v\left[-\frac{\partial w}{\partial x}e^u\sin v + \frac{\partial w}{\partial y}e^u\cos v\right]$$
from (1) & (2)
$$= \frac{\partial w}{\partial x}e^{2u}\sin v\cos v + \frac{\partial w}{\partial y}e^{2u}\sin^2 v - \frac{\partial w}{\partial x}e^{2u}\cos v\sin v + \frac{\partial w}{\partial y}e^{2u}\cos^2 v$$
$$= \frac{\partial w}{\partial y}e^{2u}(\sin^2 v + \cos^2 v)$$
$$\Rightarrow \boxed{y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v} = \frac{\partial w}{\partial y}\cdot e^{2u}}$$ Hence proved.

Ex-2: If $z = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$

prove that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$. (AKTU-2013)

Ex-3: If $z = u^2 + v^2$, $u = r\cos\theta$, $v = r\sin\theta$ (AKTU-2001, 2013).

Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$. [Ans $2r, 0$]

Ex-4: If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Case-III: **Ex-5:** If $u = f(r, s)$ and $r = x + y$, $s = x - y$

show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Case-III: If $u = f(r, s, t)$ where r, s & t are the func^n of x, y and z then

$$\boxed{\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}\end{aligned}}$$

Ex-1: If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$ (AKTU-2016, 2018)

Sol^n: Given $u = f(r, s, t)$

where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ $\Big\} \to (1)$

Now by total diff. we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{1}{y} + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}\frac{(-z)}{x^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} \longrightarrow (2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{(-x)}{y^2} + \frac{\partial u}{\partial s}\cdot\frac{1}{z} + \frac{\partial u}{\partial t}\cdot 0$$

$$\Rightarrow y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \to (3)$$

Also

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}\cdot\frac{(-y)}{z^2} + \frac{\partial u}{\partial t}\cdot\frac{1}{x}$$

$$\Rightarrow z\frac{\partial u}{\partial z} = -\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \to (4)$$

Adding eq^n (2), (3) & (4), we get

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0}$$

Ex-2: If $u = f(2x-3y, 3y-4z, 4z-2x)$ prove that
$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0 .$$ (AKTU-2015, 2014, 2019)

Sol^n: Given $u = f(2x-3y, 3y-4z, 4z-2x)$

Let
$$\left.\begin{aligned} r &= 2x-3y \\ s &= 3y-4z \\ t &= 4z-2x \end{aligned}\right\} \rightarrow (1)$$
then $u = f(r, s, t)$

Now by total differentiation,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$
$$= \frac{\partial u}{\partial r}\cdot 2 + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t}(-2)$$
$$= 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial t}$$
$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \rightarrow (2)$$
& $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$
$$= \frac{\partial u}{\partial r}\cdot(-3) + \frac{\partial u}{\partial s}\cdot 3 + \frac{\partial u}{\partial t}\cdot 0$$
$$= -3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s}$$
$$\Rightarrow \frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \rightarrow (3)$$

Also
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$
$$= \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s}\cdot(-4) + \frac{\partial u}{\partial t}\cdot 4$$
$$= -4\frac{\partial u}{\partial s} + 4\frac{\partial u}{\partial t}$$
$$\Rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \rightarrow (4)$$

Adding eq^n (2), (3) & (4), we get
$$\boxed{\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0}$$

Ex-3: If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ (AKTU-2010)

Ex-4: If $u = f(x^2+2yz, y^2+2zx)$, prove that
$$(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$$ (AKTU-2009)

Ex-5: If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ show that $x^2u_x + y^2u_y + z^2u_z = 0$.

Ex-6: If $w = \sqrt{x^2+y^2+z^2}$ and $x = u\cos v$, $y = u\sin v$, $z = uv$, then prove that
$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$$. (AKTU-2010, 2017).

Sol^n 6: Given $w = \sqrt{x^2+y^2+z^2} \rightarrow (1)$

where $x = u\cos v$, $y = u\sin v$, $z = uv$ $\} \rightarrow (2)$

from (1) & (2),
$$w = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2v^2} = \sqrt{u^2+u^2v^2}$$
$$\Rightarrow w = u\sqrt{1+v^2} \rightarrow (3)$$

Diff. eq^n ③ w.r.t u partially, we get

$$\frac{\partial w}{\partial u} = \sqrt{1+v^2} \Rightarrow u\frac{\partial w}{\partial u} = u\sqrt{1+v^2} \rightarrow ④$$

Diff eq^n ③ w.r.t v partially, we get

$$\frac{\partial w}{\partial v} = u\frac{1}{2}\times\frac{1}{\sqrt{1+v^2}}\times 2v = \frac{uv}{\sqrt{1+v^2}}$$

$$\Rightarrow v\frac{\partial w}{\partial v} = \frac{uv^2}{\sqrt{1+v^2}} \rightarrow ⑤$$

Now from ④ & ⑤, we get

$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = u\sqrt{1+v^2} - \frac{uv^2}{\sqrt{1+v^2}}$$

$$= \frac{u(1+v^2) - uv^2}{\sqrt{1+v^2}} = \frac{u}{\sqrt{1+v^2}}$$

$$\Rightarrow \boxed{u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}}$$ Hence proved.

Ex-7: If $x = u+v+w$, $y = vw + wu + uv$, $z = uvw$ and F is a func^n of x, y, z, show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

Case-IV If $f(x,y) = c$ then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

Ex-1: find $\frac{dy}{dx}$, if $x^y + y^x = a$.

Sol^n: Let $f(x,y) = x^y + y^x - a = 0 \rightarrow ①$

Then $\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$

$$\frac{\partial f}{\partial y} = x^y \log x + x y^{x-1}$$

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(yx^{y-1} + y^x \log y)}{(x^y \log x + x y^{x-1})}$$