08/11/2008

Transient State Analysis :-

$$\frac{X_o(s)}{X_i(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad \text{--- (1)}$$

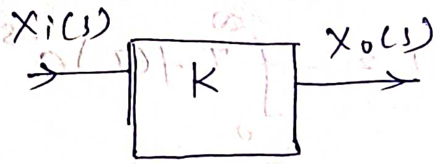
i) Zero order System :-

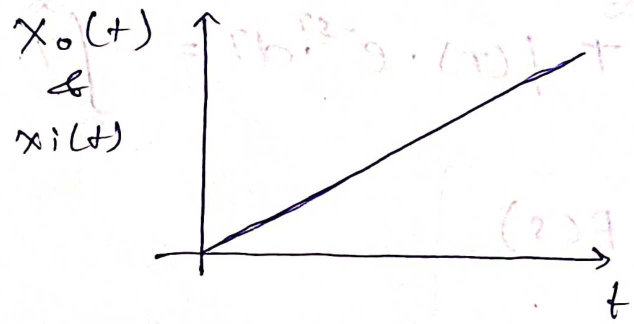
In equn. (1) if all the elements except, a_0, b_0, are zero then the resulting equn. describes as zero order system.

$$\Rightarrow \frac{X_o(s)}{X_i(s)} = \frac{b_0}{a_0}$$

let K = Gain = b_0/a_0

$$\frac{X_o(s)}{X_i(s)} = K$$





Example of zero order systems are :- Sensors & transducers.

There is no time response analysis for zero order system

as the i/p, o/p characteristics are linearly dependent.

#→ In closed loop control configuration the F/B (feedback) elements $H(s)$ represents zero order system.

#→ 2): First order System:

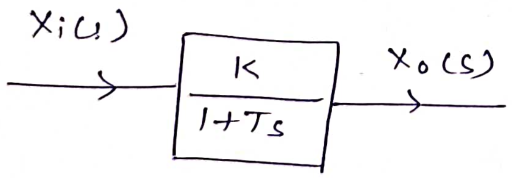
#→ In equn. ① if all the terms except a_1, a_0, b_0 are zero then resulting equn. describe first order system.

$$\Rightarrow \frac{X_o(s)}{X_i(s)} = \frac{b_0}{a_1 s + a_0}$$

$$= \frac{b_0/a_0}{\frac{a_1}{a_0} s + 1}$$

$$\therefore \boxed{\begin{aligned} K &= \text{Gain} = b_0/a_0 \\ T &= \text{Time constant elements} = \frac{a_1}{a_0} \end{aligned}}$$

$$\frac{X_o(s)}{X_i(s)} = \frac{K}{1+Ts}$$



#→ Example of Ist order system = RC-filter

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{RCs+1}$$

$$\boxed{T = RC}$$ → time constant.

Transient Analysis:-

let $X_i(s) = \frac{1}{s}$ (unit step i/p)

$$X_o(s) = \frac{K}{S(1+TS)}$$

$$X_o(s) = K\left[\frac{1}{S} - \frac{T}{1+TS}\right]$$

$$X_o(s) = K\left[\frac{1}{S} - \frac{1}{S+1/T}\right]$$

$$X_o(t) = K[\underbrace{1}_{S.S} - \underbrace{e^{-t/T}}_{\text{Transient state}}]$$

As $t \to \infty$

$X_o(t) \to 1$

$\Rightarrow \lim_{t\to\infty} X_o(t) = 1$

$$\lim_{t\to\infty} K\left[1 - e^{-t/T}\right] = 1$$

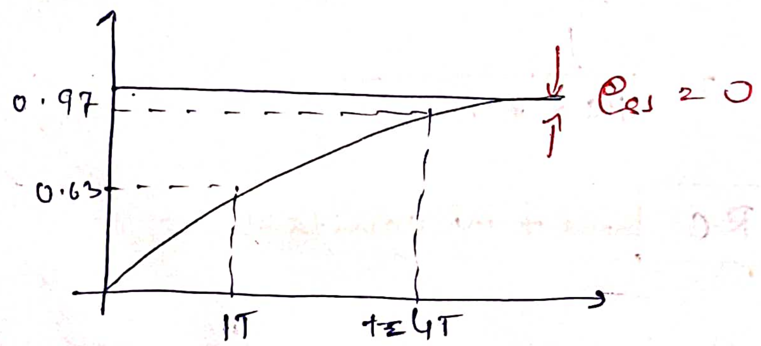
$\boxed{K=1}$ → Gain of I^{st} order system is "1".

$$X_o(t) = 1 - e^{-t/T}$$

At $t = T$

$$X_o(t) = 1 - e^{-1}$$

$$= 1 - 0.37 = 0.63$$



Now $e(t) = X_i(t) - X_o(t)$

$$= 1 - [1 - e^{-t/T}]$$

$$e(t) = e^{-t/T}$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$= \lim_{t \to \infty} e^{-t/T} = 0$$

3):- Second Order Systems :-

#→ Response of the second order system exhibits continuous and sustain oscillations about the steady state value of i/p, with the freq. known as "Undamped Natural freq" (ω_n)".

#→ This oscillations in the response ~~are~~ are damped to the steady state of value of i/p using different damping method.

#→ The damping is mathematically ~~mathematical~~ represented as a damping factor "Geta" (G).

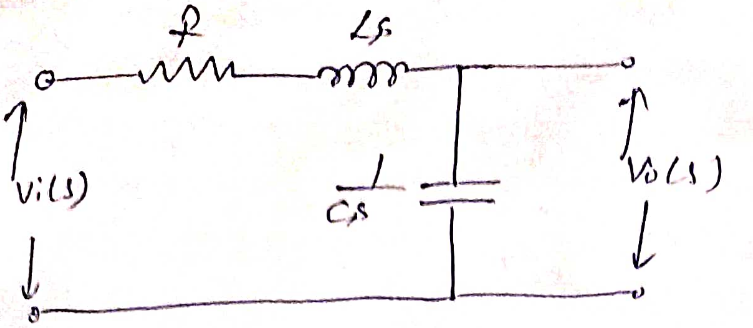
#→ Therefore the transfer function of 2nd order system is expressed in terms of Geta and ω_n as

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2G\omega_n s + \omega_n^2}}$$

#→ All indicating instruments (moving ~~gal~~ coil, ~~ga~~ moving iron, electrodynamo meter) etc are example of 2nd order system.

Another Example of ~~a~~ 2nd order System is

R-L-C - N/w.



$$V_i(s) = I(s)\left[R + Ls + \frac{1}{Cs}\right]$$

$$V_i(s) = I(s)\left[\frac{LCs^2 + RCs + 1}{Cs}\right]$$

$$V_o(s) = I(s) \cdot \frac{1}{Cs}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$s^2 + \frac{R}{L} + \frac{1}{LC} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\boxed{\omega_n = \frac{1}{\sqrt{LC}} \ r/s}$$

$$2\zeta\sqrt{\frac{1}{LC}} = \frac{R}{L}$$

$$\boxed{\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}}$$

Ex:- Mass-damper spring system :-

$$F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

$$F(s) = [Ms^2 + Bs + K]\,X(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} = \frac{\frac{1}{M}}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

$$s^2 + \frac{B}{M}s + \frac{K}{M} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\boxed{\omega_n = \sqrt{\frac{K}{M}}\ r/s}$$

And $2 \times \zeta \times \sqrt{\frac{K}{M}} = \frac{B}{M}$

$$\boxed{\zeta = \frac{B}{2\sqrt{KM}}}$$

here $\{\zeta \to B \to R\}$

Example :- Another ex. of 2nd order system is Position Control system :..

So,

from B.D of Position Control system (on previous pages)

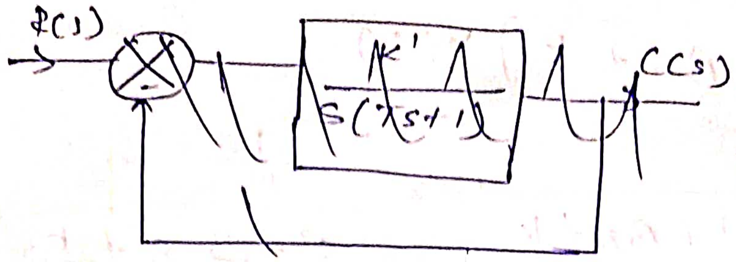
$$\frac{\frac{K_T}{R_a[Js^2 + f_0 s]}}{1 + \frac{K_T K_b s}{R_a(Js^2 + f_0 s)}}$$

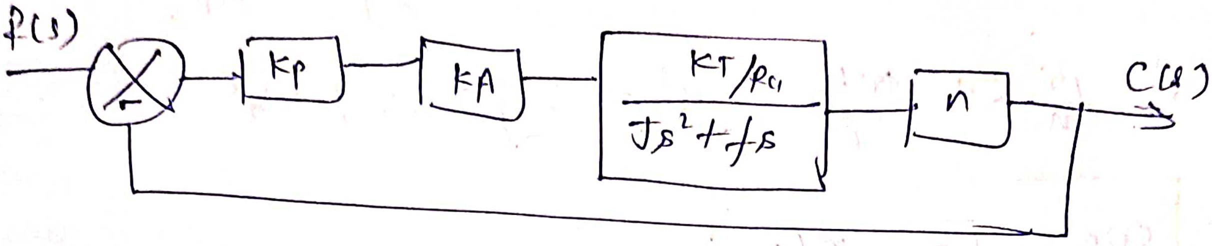
$$\Rightarrow \frac{K_T}{R_a(Js^2 + f_0 s) + K_T K_b s}$$

$$\Rightarrow \frac{K_T/R_a}{Js^2 + \underbrace{f_0 s + \frac{K_T K_b s}{R_a}}_{fs}}$$

$$\therefore \Rightarrow \frac{K_T/R_a}{Js^2 + fs}$$

So, the B.D ~~simj~~ simplifies as.





$$\frac{\left\{\frac{K_p K_A K_T \cdot n}{R_a}\right\} \Rightarrow K}{Js^2+fs}$$

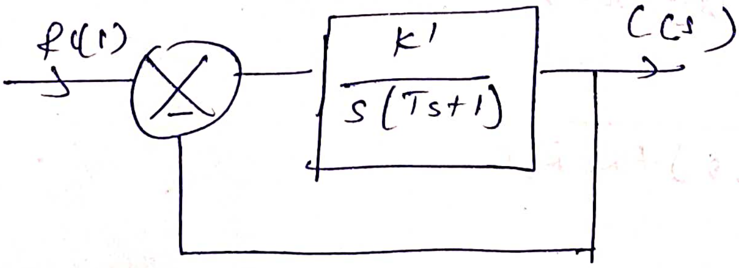
$$\Rightarrow \frac{K}{s[Js+f]} = \frac{K/f}{s\left[\frac{J}{f}s+1\right]}$$

let $K' = K/f$

$T = J/f$

$$\Rightarrow \frac{K'}{s(Ts+1)}$$

$\Rightarrow$ B.D is simplifies as



$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K'}{s(Ts+1)}}{1+\frac{K'}{s(Ts+1)}} \Rightarrow \frac{K'}{Ts^2+s+K'}$$

$\therefore \frac{C(s)}{R(s)} = \frac{K'/T}{s^2 + \frac{1}{T}s + \frac{K'}{T}}$

$2 \times \zeta \times \sqrt{\frac{K'}{T}} = \frac{1}{T}$

$\boxed{\zeta = \frac{1}{2\sqrt{K'T}}}$

$\omega_n^2 = \frac{K'}{T} \Rightarrow \boxed{\omega_n = \sqrt{\frac{K'}{T}}\ r/s}$

→ Effect of Damping on Closed loop poles and Nature of Response :-

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for C.L poles :-

$1 + G(s)H(s) = 0$

$\Rightarrow s^2 + 2\zeta\omega_n + \omega_n^2 = 0$

$\Rightarrow$ Roots of the equn. are ,,,

$$\frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$\Rightarrow -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$\Rightarrow D = \zeta^2 - 1 = 0 \Rightarrow \zeta = 1$

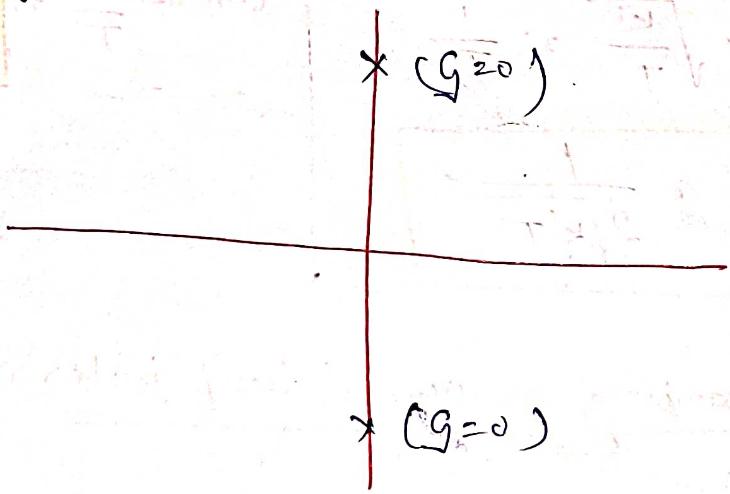
$D = \zeta^2 - 1 < 0 \Rightarrow \zeta < 1$

and $D = \zeta^2 - 1 > 0 \Rightarrow \zeta > 1$

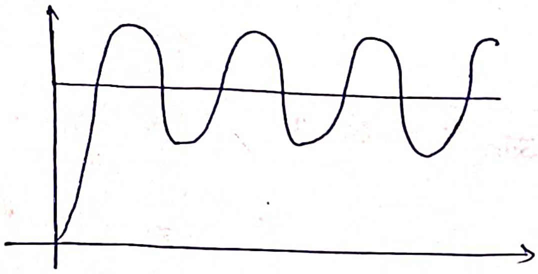
Now, from above analysis we get 4- cases, those are:-

Case 1:- Undamped case

$(\zeta = 0)$



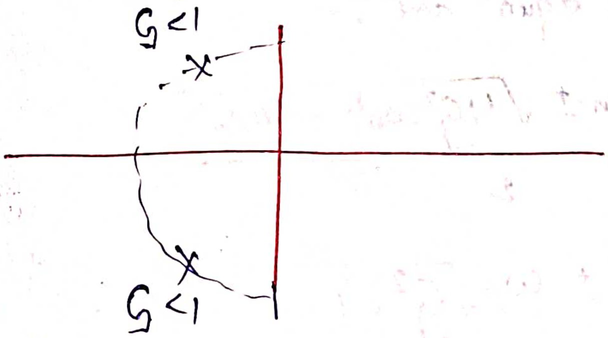
Response :- $(\sin \omega t)$



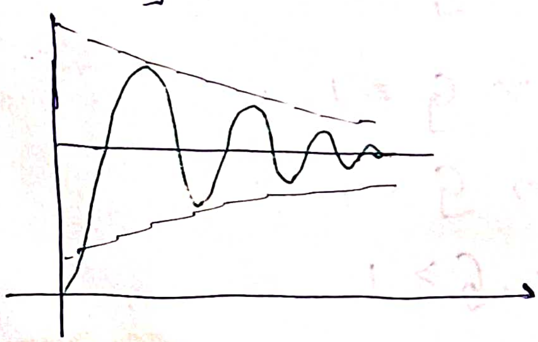
$\boxed{\text{with freq} = \omega_n \ r/s}$

Case 2:- Underdamped condition

i.e $\zeta < 1$ $\boxed{1 > \zeta > 0}$

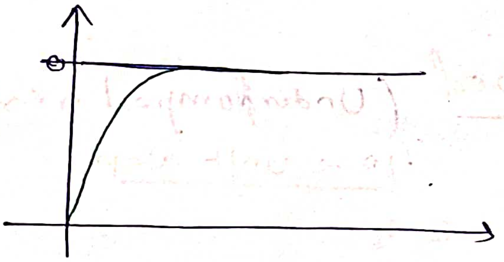


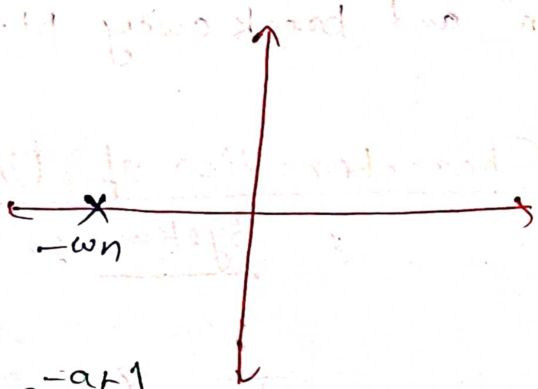
Response:- $(e^{-at} \sin \omega t)$



Case:-3:- Critically damped condition

i.e ($\zeta = 1$)

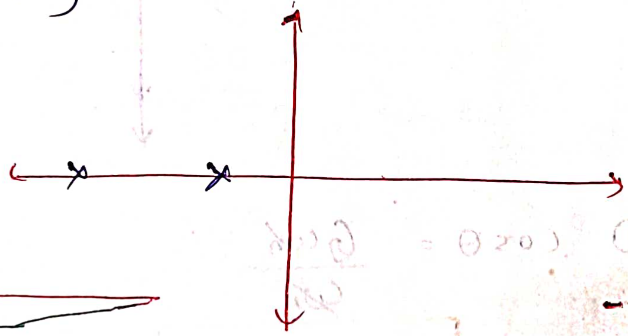


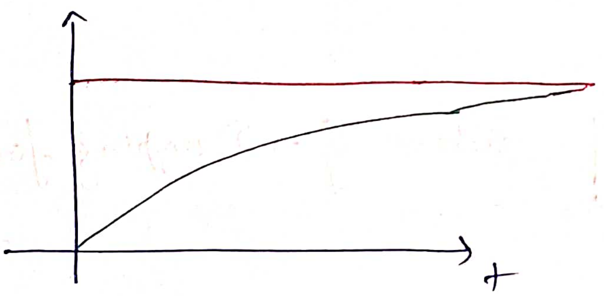


Response:- (te^{-at})

Case:-4:- Overdamped Condition

i.e ($\zeta > 1$)





Response:- ($e^{-at} + e^{-bt}$)

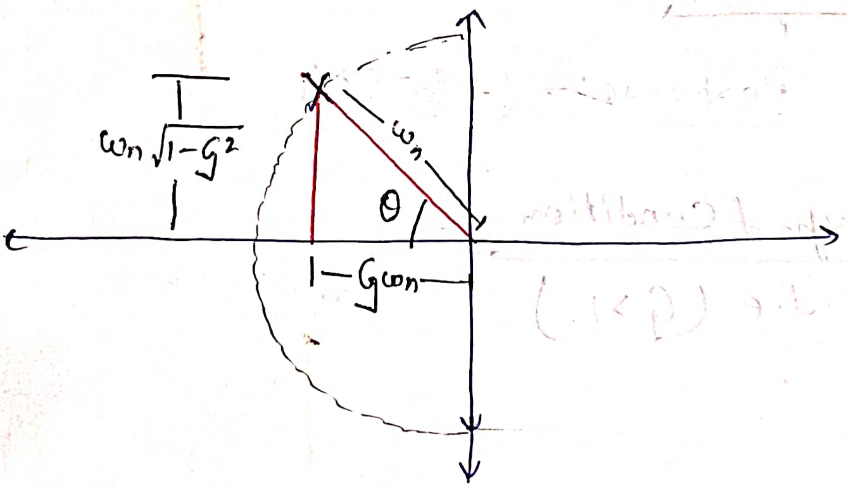
#→ Most of the control system are design for the value of damping less than "1", because response is neither too fast ($\zeta = 1$) nor too slow or slogisy ($\zeta > 1$), in reaching the steady state value of the i/p.

#→ The underdamped response can be analyse with more no. of performance indices

#→ The root locus of 2nd order system obtain by varying damping factor "ζ" from 0 to ∞, is semicircular path with radius "ω_n" and break away pt. at "$-\omega_n$" on the $-ve$ real axis.

Characteristics of Underdamped System

(Underdamped response)
i/p → unit step



① $\cos\theta = \frac{\zeta \cancel{\omega_n}}{\cancel{\omega_n}}$

$\Rightarrow \boxed{\theta = \cos^{-1}\zeta}$ where $\zeta \to$ Damping factor.

②:- Damping coefficient

$$\boxed{\alpha = \zeta\omega_n}$$

③:- Damped Natural freq.

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2} \quad r/s}$$

④:- Time Constant of ~~Underf~~ Underdamped response $\Big\} \rightarrow \boxed{T = \frac{1}{\alpha} = \frac{1}{\zeta\omega_n}}$

Transient Response:-

Let $R(s) = \frac{1}{s}$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \frac{1}{s} - \frac{(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{s^2+2\zeta\omega_n s+\omega_n^2} - \frac{\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\omega_d}{(s^2+2\zeta\omega_n s+\omega_n^2)} \times \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}$$

Now taking laplace inverse of above eqn. $\left(\text{as } \omega_d = \omega_n\sqrt{1-\zeta^2}\right)$

$$\Rightarrow C(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - e^{-\zeta\omega_n t}\sin\omega_d t \cdot \frac{\zeta}{\sqrt{1-\zeta^2}}$$

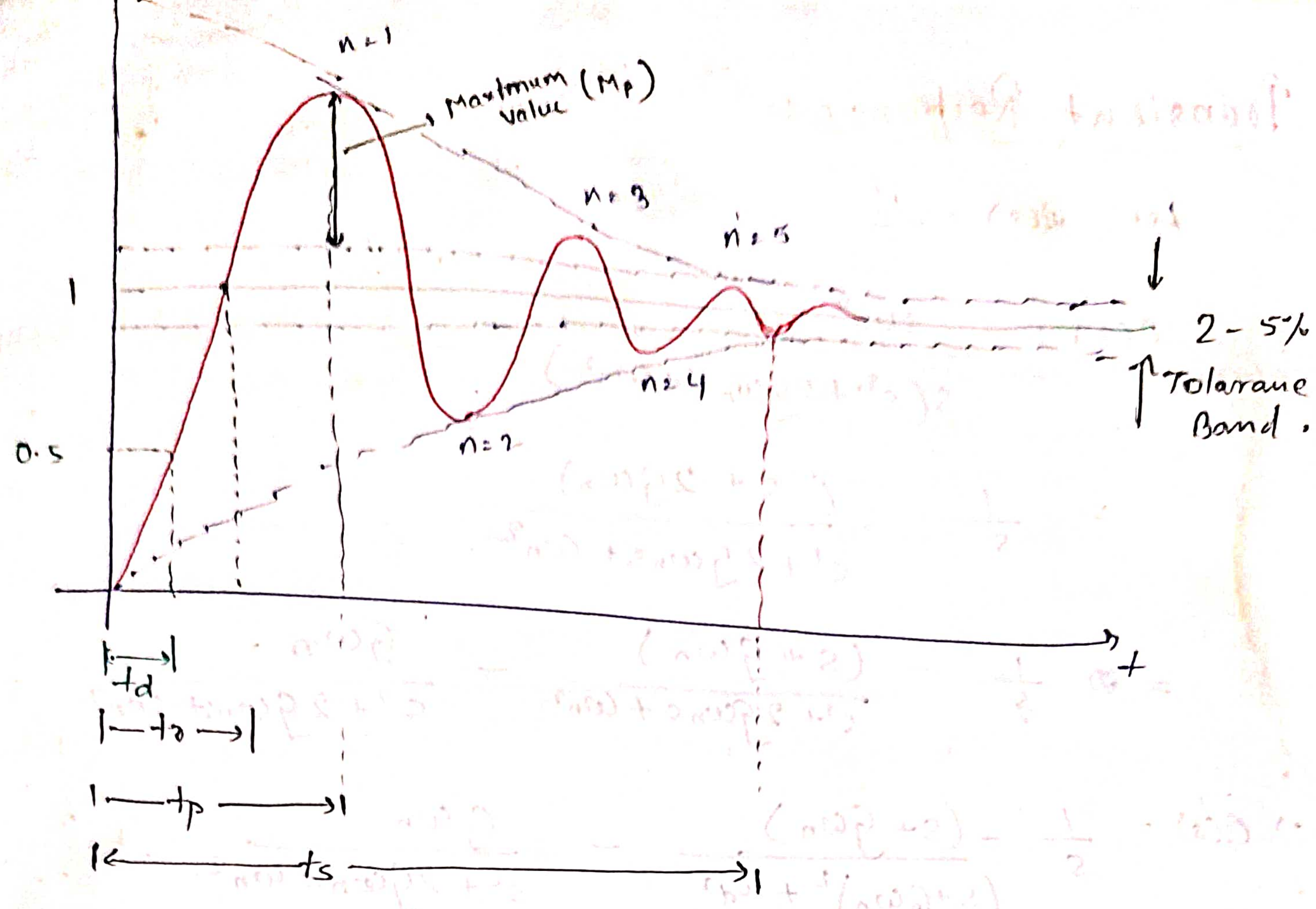
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

Now as we know

$$\boxed{A\sin\omega t + B\cos\omega t = \sqrt{A^2+B^2}\cdot\sin\left(\omega t + \tan^{-1}B/A\right)}$$

By applying this formula

$$C(t) = \underbrace{1}_{S.S} - \underbrace{\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_d t + \tan^{-1}\left\{\frac{\sqrt{1-\zeta^2}}{\zeta}\right\}\right]}_{\text{Transient}}$$



(1):- Delay time:- $t_d = \frac{1 + 0.7\zeta}{\omega_n}$ (secs)

(2):- Rise time :- $t_r = \frac{\pi - \theta}{\omega_d}$ (secs) $\left[\text{where:, } \theta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right]$ (radians)

(3):- ~~Settle~~ Peak time:- $t_p = \frac{n\pi}{\omega_d}$

(4):- Settling time :- $t_s =$ 2% T.B $\rightarrow 4T \rightarrow 4/\zeta\omega_n$

5% T.B $\rightarrow 3T \rightarrow \frac{3}{\zeta\omega_n}$

(5): Maxm peak overshoot:- $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$